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ERRATA in Vol. 31, No. 2, Nov.-Dec., 1957: In some of the magazines, the figures on pages 63 and 91 should be interchanged.

PRINCIPLE OF INDUCTION AND A SEQUENCE OF GENEROSITIES

(Popular Article)

A. R. Amir-Moez

The Shah came out of the court followed by his forty ministers. A beggar bowed. The King put a penny in the beggar's hand and said to his ministers, "Come here according to your rank and give him twice as much as you see in his hand."

1) Find the general term in this **sequence**.

2) How much does the beggar get?

To get the answer to part 2 of this problem, we may spend enough time and use our addition and multiplication tables. But we would like to explain the principle of induction in a very elementary way, and make use of it in solving the problem.

I. Let n denote a natural number (whole number) such as 5, 6, ... If a property whose truth for n implies its truth for $n+1$, is true for a number, for example 5, then this property is true for all natural numbers larger than 5. This is indeed common sense to say: a property which if true for any whole number is true for the next whole number and is true for 5 clearly it will be true for 6. The property being true for 6 it has to be true for 7, and this goes on for all natural numbers larger than 5.

II. We give a very simple example to demonstrate I. Suppose we would like to add all the whole numbers from 5 to 237. One way of doing this, is taking time and adding them. But we shall use a more interesting way of doing the problem.

We observe that:

$$5 + 6 \quad = 2\left(\frac{5+6}{2}\right) = 11$$

$$5 + 6 + 7 = 3\left(\frac{5+7}{2}\right) = 18$$

(continued on page 132)

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$$5 + 6 + 7 + 8 = 4\left(\frac{5+8}{2}\right) = 26$$

This process suggests that for finding this sort of sum we have to multiply the average of the first and the last by the number of them. For example, for adding whole numbers from 5 to 237, including 237, we get $233\left(\frac{5+237}{2}\right) = 28193$. I.e. there are 233 terms in $5 + 6 + 7 + \dots + 237$, and the average of 5 and 237 is $\frac{5+237}{2}$.

III. Now we concentrate on the sequence of generosities.

The Shah gives 1¢; then the beggar has 1¢,
1st minister gives 2¢; then the beggar has 3¢,
2nd minister gives 6¢; then the beggar has 9¢,
3rd minister gives 18¢; then the beggar has 27¢,
4th minister gives 54¢; then the beggar has 81¢, etc.

Comparing these terms we see that each term is three times as great as the previous one. We can write the terms of the sequence as follows:

After the Shah's contribution the beggar has 1¢
After the 1st minister's contribution the beggar has 3^1¢
After the 2nd minister's contribution the beggar has 3^2¢
After the 3rd minister's contribution the beggar has 3^3¢
After the 4th minister's contribution the beggar has 3^4¢
....

The principle of induction suggests that after the nth minister's contribution the beggar has 3^n¢. This answers the first part of the problem. For the second part we have to see how much the beggar shall have after the 40th minister's contribution. Clearly this is 3^{40}, and $3^{40} = 81^{10} = (6561)^5 =$

$$12157665490569288801.$$

Therefore the beggar gets

$$\$12,157,665,490,569,288.01.$$

He can run the country now. Can't he?

Queens College

A SYMBOLIC METHOD FOR FINDING INTEGRALS OF LINEAR DIFFERENCE AND DIFFERENTIAL-DIFFERENCE EQUATIONS

K. L. Cooke

The use of symbolic operators in finding particular solutions of linear, non-homogeneous differential equations with constant coefficients is common in textbooks on differential equations. Analogous methods have long been known for difference equations of the form

$$\sum_{i=0}^{n} a_i \mathrm{u}(x+i) = \mathrm{f}(x),$$

in which the "spans" are integers.[1] The purpose of the present note is to show that the same methods may be applied to difference equations with arbitrary real spans and to differential-difference equations. The presentation will be brief, since the methods have been widely discussed in the simpler cases.

The differential-difference equations to be considered have the form

$$\sum_{j=0}^{n} \sum_{i=0}^{m} a_{ij} \mathrm{u}^{(j)}(x+b_i) = \mathrm{f}(x), \tag{1}$$

where x is a real variable, $\mathrm{u}^{(0)}(x) \equiv \mathrm{u}(x)$, the a_{ij} are real constants, $0 = b_0 < b_1 < \dots < b_m$, and $\mathrm{f}(x)$ is a given real function. In case $n = 0$, (1) becomes a pure difference equation, and in case $m = 0$, it becomes a pure differential equation. The method outlined below applies in these cases as well as in the general case. As usual with a linear equation, the general solution of (1) consists of the sum of the general solution of the homogeneous equation

$$\sum_{j=0}^{n} \sum_{i=0}^{m} a_{ij} \mathrm{u}^{(j)}(x+b_i) = 0, \tag{2}$$

and any particular solution of (1). In this note we shall say nothing about methods for solving (2), referring the reader to a paper of Wright, [4], in which a complete solution is given. In fact, this note is restricted to the problem of finding one particular solution of (1), by the easiest method

[1]For example, see Boole, [1], pp. 212-219, or Nörlund, [3], pp. 398-400.

possible, in case f(x) is an exponential function, a sinusoidal function, a polynomial, or a combination of these. More general methods for finding particular solutions, such as variation of parameters, are known, but, just as for pure differential equations, they do not provide as easy a solution when f(x) has the special forms mentioned.

The symbolic operators to be used here are the familiar D, Δ, and E, defined by the relations

$$\mathrm{D}u(x) = u'(x),$$
$$\Delta u(x) = u(x+1) - u(x),$$
$$\mathrm{E}u(x) = u(x+1).$$

For any positive number h and any real constant k we also define

$$(k\mathrm{E})^h u(x) = k^h u(x+h)$$

and

$$(kD)u(x) = k(Du(x)), \qquad (D = \mathrm{D} \text{ or } \Delta).$$

Sums and products of operators have the customary meaning; for example

$$(\mathrm{D}+\Delta)u(x) = \mathrm{D}u(x) + \Delta u(x),$$
$$(\mathrm{D}\Delta)u(x) = \mathrm{D}(\Delta u(x)), \quad \text{etc.}$$

With these definitions, the commutative, associative, and distributive laws hold, so that these operators may be manipulated in the usual algebraic ways.[1] In particular note that

$$\mathrm{E}^{h_1}\mathrm{E}^{h_2}u(x) = \mathrm{E}^{h_1+h_2}u(x) = u(x+h_1+h_2).$$

Another extremely important relation is

$$\mathrm{E}^h = (1+\Delta)^h = e^{h\mathrm{D}}, \tag{3}$$

where e^{hD} is used to represent the series

$$1 + h\mathrm{D} + \frac{h^2\mathrm{D}^2}{2!} + \dots .$$

[1]For a more complete discussion, refer to Ford, [2], p. 196.

It is clear that $E = 1 + \Delta$. To verify that $E^h = e^{hD}$, note that

$$E^h f(x) = f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(x)}{2!} + \ldots$$

provided the Taylor's series converges and represents $f(x+h)$. This will be true for every x and every h for an exponential or sinusoidal function or a polynomial. Thus (3) will be valid at least for the class of functions we are admitting into the present discussion.

We can now examine the effect of operating on elementary functions with these operators. First of all, let F(D) represent the symbolic operator

$$F(D) = \sum_{j=0}^{n} \sum_{i=0}^{m} a_{ij} E^{b_i} D^j = \sum_{j=0}^{n} \sum_{i=0}^{m} a_{ij} e^{b_i D} D^j .$$

The latter sum may be regarded as an abbreviation for the first in those cases in which $E \neq e^D$. Equation (1) may be written symbolically in the form

$$(1') \qquad F(D)u(x) = f(x).$$

We first point out that

$$(4) \qquad F(D)(ke^{cx}) = F(c)ke^{cx}.$$

For

$$F(D)(ke^{cx}) = k\sum\sum a_{ij} E^{b_i} D^j (e^{cx}) = ke^{cx}\sum\sum a_{ij} c^j e^{cb_i}.$$

(The summations are over the same ranges as before.) Secondly we observe that

$$(5) \qquad F(D)\left\{e^{cx}f(x)\right\} = e^{cx}\left\{F(D+c)f(x)\right\}.$$

To prove this, we use the well-known[1] relation

$$D^j\left\{e^{cx}g(x)\right\} = e^{cx}(D+c)^j g(x).$$

From this relation we find

$$\begin{aligned} E^{b_i}D^j\left\{e^{cx}f(x)\right\} &= D^j\left\{e^{c(x+b_i)}f(x+b_i)\right\} \\ &= e^{c(x+b_i)}(D+c)^j f(x+b_i) \\ &= e^{c(x+b_i)}(D+c)^j\left\{E^{b_i}f(x)\right\} \\ &= e^{cx}\left\{(D+c)^j e^{b_i(D+c)}f(x)\right\}. \end{aligned}$$

(5) follows at once. Both (4) and (5) have a clear symbolic meaning even for those functions $f(x)$ for which (3) is not valid.

[1] Ford, [2], p. 95.

Now let the symbol

$$F^{-1}(D)f(x) = \left(\sum\sum a_{ij}E^{b_i}D^j\right)^{-1}f(x)$$

denote any particular solution of equation (1). Note that

$$F(D)\left\{F^{-1}(D)f(x)\right\} = f(x). \tag{6}$$

The following theorems show how to calculate $F^{-1}(D)f(x)$ for functions $f(x)$ of the special types already described.

Theorem 1.

$$F^{-1}(D)(ke^{cx}) = ke^{cx}/F(c),$$

provided $F(c) \neq 0$.

To prove this, observe that by (4),

$$F(D)\left\{ke^{cx}/F(c)\right\} = F(c)ke^{cx}/F(c) = ke^{cx}.$$

Theorem 2.

$$F^{-1}(D)\left\{e^{cx}f(x)\right\} = e^{cx}F^{-1}(D+c)f(x).$$

To prove this, notice that

$$F(D)\left\{e^{cx}F^{-1}(D+c)f(x)\right\} = e^{cx}F(D+c)\left\{F^{-1}(D+c)f(x)\right\},$$

by (5). By (6), the expression on the right is $e^{cx}f(x)$.

Theorem 3.

$$F^{-1}(D)\left\{e^{cx}\sin\alpha x\right\} = Im\left\{F^{-1}(D)e^{(c+\alpha j)x}\right\},$$

where Im denotes the imaginary part of.

Theorem 4.

$$D^{-1}f(x) = \int f(x)dx + C.$$

Theorem 5. *In order to evaluate* $F^{-1}(D)P_n(x)$, *where* $P_n(x)$ *is a polynomial of degree* n, *let* $F(D)$ *be formally expanded in ascending powers of* D. *Let the result be*

$$p_0D^r(1 + p_1D + \dots).$$

Let

$$(1 + p_1 D + \dots)^{-1}$$

have the formal expansion

$$1 + q_1 D + q_2 D^2 + \dots$$

in ascending powers of D. *Then*

$$\text{(7)} \qquad F^{-1}(D)P_n(x) = p_0^{-1} D^{-r}\left\{(1 + q_1 D + \dots + q_n D^n)P_n(x)\right\}.$$

Proof. We must show that F(D) operating on the right member of (7) yields $P_n(x)$, or in other words that

$$p_0 D^r(1 + p_1 D + \dots)p_0^{-1} D^{-r}\left\{(1 + q_1 D + \dots + q_n D^n)P_n(x)\right\} = P_n(x).$$

Since we are dealing only with polynomials, there is no question of convergence. By the commutativity of our operators and by (6), the above is equivalent to proving that

$$\text{(8)} \qquad (1 + p_1 D + \dots)(1 + q_1 D + \dots + q_n D^n)P_n(x) = P_n(x).$$

However,

$$\text{(9)} \qquad (1 + p_1 D + \dots)(1 + q_1 D + \dots + q_n D^n + \dots) = 1,$$

when interpreted as a Cauchy product, is an identity, and

$$\text{(10)} \qquad (1 + p_1 D + \dots)(q_{n+1} D^{n+1} + \dots)P_n(x) = 0$$

since $P_n(x)$ is of degree n. (8) thus follows from (9) and (10), and the proof is complete.

We shall conclude this paper with two examples. First consider the equation

$$u'(x+1) - 2u(x+1) + u'(x) - 2u(x) = e^x.$$

By Theorem 1, a particular solution is

$$u(x) = \frac{e^x}{e-2e+1-2} = -\frac{e^x}{e+1}.$$

As a second example, consider

$$F(D)u(x) \equiv (De^D - 2e^D + D - 2)u(x) = e^{2x}.$$

In this case, $F(2) = 0$, and Theorem 1 cannot be directly applied. Instead, we use Theorems 2 and 5, as follows:

$$u(x) = (De^{D} - 2e^{D} + D - 2)^{-1}e^{2x} = e^{2x}(De^{D+2} + D)^{-1}(1).$$

But

$$\left\{D(e^{2}e^{D} + 1)\right\}^{-1}(1) = D^{-1}\left\{(1 + e^{2}) + e^{2}D + \dots\right\}^{-1}(1)$$
$$= (1 + e^{2})^{-1}D^{-1}\left\{1 - \frac{e^{2}}{1+e^{2}}D + \dots\right\}(1)$$
$$= (1 + e^{2})^{-1}D^{-1}(1) = (1 + e^{2})^{-1}x$$

Hence

$$u(x) = \frac{xe^{2x}}{1+e^{2}}$$

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ORTHOGONAL TETRAHEDRON

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An Orthogonal Tetrahedron is one whose altitudes concur at its orthocentre where also* concur its bi-altitudes such that the pairs of its opposite edges are orthogonal. One of the three conditions stated implies the other two. The aim of this paper is to assume the concurrency of its bi-altitudes and thence deduce the other two. Keeping this aim in view the problem is reduced to one on conics in a plane when we translate the perpendicularity of any two elements in space to the conjugacy of their traces in the plane at infinity with respect to the circle or an absolute conic there. The treatment is analytic thereafter, leading to very interesting algebraic identities.

1. *Introduction.*

Let ABCD be a Tetrahedron T, and LL′, MM′, NN′ be the shortest distances of, or its bi-altitudes to, the pairs of its opposite edges AD, BC; BD, CA; CD, AB respectively (figure 1); let (ad), (bc), ... (ll′), ... be the traces (that means traces at infinity) of the lines AD, BC, ... LL′, ... , ... and C_i be the circle or Absolute conic at infinity such that the conjugacy with respect to C_i of the traces of any two elements in space implies their perpendicularity, e.g. if AD & BC are at right angles, (ad) & (bc) are conjugate for C_i and vice-versa. The six traces of the edges of T form the vertices of the Quadrilateral q formed by the traces of its four faces; let PQR be the diagonal triangle of q (figure 2).

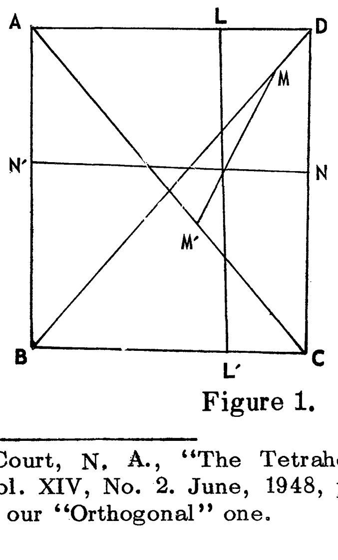

Figure 1.

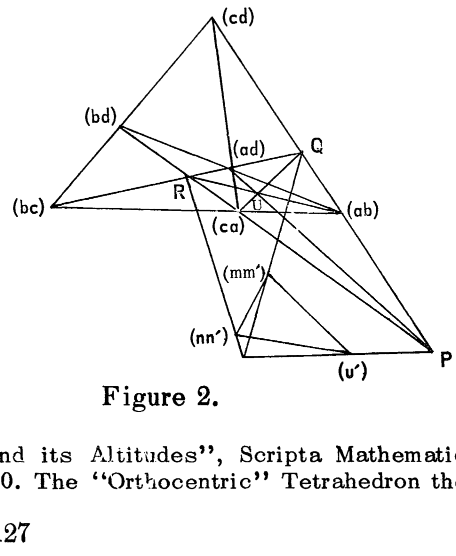

Figure 2.

*Court, N. A., "The Tetrahedron and its Altitudes", Scripta Mathematica, Vol. XIV, No. 2. June, 1948, pp. 85-90. The "Orthocentric" Tetrahedron there is our "Orthogonal" one.

The triangles PQR and (ad) (ab) (ca) are seen to be in perspective, for their corresponding sides meet in three collinear points, viz. (bc), (cd), (bd); the joins of their corresponding vertices then concur, say at U.

LL′ being perpendicular to both AD & BC, (ll′) is then conjugate to both (ad) & (bc) or is the pole of their join QR with respect to C_i. Similarly (mm′) & (nn′) are the poles of RP & PQ respectively with respect to C_i. The two triangles PQR & (ll′) (mm′) (nn′) are thus polar reciprocals for C_i and therefore * in perspective; let their centre of perspectivity be 0.

2. *Problem on Conics.*

If NN′ intersects MM′, it is a generator of the quadric Q_1 determined by AB, MM′, CD & BD, CA. If LL′ intersects NN′, it is a generator of the quadric Q_2 determined by BC, NN′, AD & CD, AB. If MM′ intersects LL′, it is a generator of the quadric Q_3 determined by CA, LL′, BD & AD, BC. The traces of these three quadrics Q_i $(i = 1, 2, 3)$ are then three conics q_i each passing through six points, viz. (bd), (ca), (cd), (ab), (mm′), (nn′); (cd), (ab), (ad), (bc), (nn′), (ll′); (ad), (bc), (bd), (ca), (ll′), (mm′) respectively. *The concurrency of the bi-altitudes of T thus implies the existence of these three conics q_i from which we wish to deduce the conjugacy of the three pairs of opposite vertices of q for C_i, i.e. the orthogonality of the three pairs of opposite edges of T and thence the orthogonality of T itself. The problem now before us is then one on conics in a plane, i.e. "If every two pairs of opposite vertices of a quadrilateral and the poles of their joins with respect to a conic C_i lie on a conic, the existence of the three such conics implies the self-conjugacy* of the quadrilateral for the conic C_i."*

3. *Solution of the Problem.*

a. Let the plane of the problem be t = 0, PQR (Art. 1) be the reference triangle there, and U, its *unit** point. The co-ordinates of the vertices of the quadrilateral q, in pairs, are then $(0, 1, 1), (0, 1, -1)$; $(1, 0, 1), (1, 0, -1)$; $(1, 1, 0)$, $(1, -1, 0)$. If 0 be (f, g, h); (ll′), (mm′), (nn′) can then be taken as $(f+u, g, h)$, $(f, g+v, h)$, $(f, g, h+w)$ respectively. The equation of the conic C_i, for which the triangles PQR & (ll′) (mm′) (nn′) are polar reciprocal, is found to be

$$\Sigma gh(vw + gw + vh)x^2 - 2fgh\, \Sigma uyz = 0.$$

b. $x^2 - y^2 - z^2 + 2Fyz = 0$ is seen to be the equation of the pencil of conics through the four points (bd), (ca), (cd), (ab) with F as parameter; it becomes q_1 if it passes through (mm′) & (nn′) both, i.e. if $f^2 - (g+v)^2 - h^2 + 2F(g+v)h = 0 = f^2 - g^2 - (h+w)^2 + 2Fg(h+w)$, or eliminating F, $g(h+w)$

*Baker, E. F., *Introduction to Plane Geomary,* pp. 191, 204, 85.

$(f^2-g^2-h^2-v^2-2gv) = h(g+v)(f^2-g^2-h^2-w^2-2hw)$, or, $(gw-hv)(f^2-g^2-h^2)-gh(v^2-w^2+2gv-2hw)-vw(gv-hw+2g^2-2h^2) = 0$, or,

$$(A) \equiv f^2(gw-hv) - (g^2-h^2)(gw+hv+2vw) - gh(v^2-w^2) - vw(gv-hw) = 0.$$

Similarly the conditions for q_2 & q_3 to exist respectively are

$$(B) \equiv g^2(hu-fw) - (h^2-f^2)(hu+fw+2wu) - hf(w^2-u^2) - wu(hw-fu) = 0,$$

$$(C) \equiv h^2(fv-gu) - (f^2-g^2)(fv+gu+2uv) - fg(u^2-v^2) - uv(fu-gv) = 0.$$

c. Again q will be self-conjugate for C_i, if its pairs of opposite vertices form pairs of conjugate points for C_i; the conditions for the same are found to be

$$(A') \equiv u(g^2-h^2) + (u+f)(gv-hw) = 0,$$

$$(B') \equiv v(h^2-f^2) + (v+g)(hw-fu) = 0,$$

$$(C') \equiv w(f^2-g^2) + (w+h)(fu-gv) = 0.$$

Two of them are independent; the third follows from the two, for

$$(D') \equiv f(A') + g(B') + h(C') = 0.$$

d. Now we may observe the following relations that exist between (A), (B), (C) on one side and (A′), (B′), (C′) on the other to deduce the self-conjugacy of q for C_i from the existence of the three conics $q_i (i = 1, 2, 3)$.

$$(A) \equiv (h+w)(B') + (g+v)(C') \;\ldots\ldots\ldots\ldots\ldots\; (i),$$

$$(B) \equiv (f+u)(C') + (h+w)(A') \;\ldots\ldots\ldots\ldots\ldots\; (ii),$$

$$(C) \equiv (g+v)(A') + (f+u)(B') \;\ldots\ldots\ldots\ldots\ldots\; (iii).$$

Eliminating (C′) from the identity (D′) and (ii) we have

$$h(B) \equiv (h^2-f^2+hw-fu)(A') - g(f+u)(B') \;\ldots\ldots\ldots\; (iv).$$

Eliminating (B′) from (iii) & (iv), we get

$$h(B) + g(C) \equiv (g^2+h^2-f^2+gv+hw-fu)(A') \;\ldots\ldots\; (v).$$

Similarly we may have

$$f(C) + h(A) \equiv (h^2+f^2-g^2+hw+fu-gv)(B') \dots\dots\dots\dots (vi),$$

$$g(A) + f(B) \equiv (f^2+g^2-h^2+fu+gv-hw)(C') \dots\dots\dots\dots (vii).$$

e. Incidently we have achieved much more than the solution of the problem before us. Let us now visualize the same through the interpretation of the identities arrived at above.

(D′), when interpreted in space, tells that: *If two pairs of opposite edges of a tetrahedron be orthogonal, the third pair too is orthogonal and hence the tetrahedron is orthogonal.* It is too well known a result otherwise too, but we record it to make use of hereafter.

The first evident interpretation of (i), (ii) and (iii) is: *If any two pairs of opposite vertices of q are conjugate for C_i, all the three conics q_i exist, i.e. if any two pairs of opposite edges of T are orthogonal, its bi-altitudes concur.* But this being a known result and not our aim too, we pass on to the other interpretation of theirs: *If one of the three conics q_i exists and a pair of opposite vertices of q be conjugate for C_i, another pair too is conjugate for C_i and hence q is self-conjugate for C_i.* I.e. *If two bi-altitudes of a semi-orthocentric* tetrahedron meet, a second pair of its opposite edges is orthogonal and hence the tetrahedron is orthogonal.*

(v), (vi) & (vii), when interpreted give the solution of the problem as follows: *If two of the three conics q_i exist, a pair of opposite vertices of q are conjugate for C_i; again one of the two q_i considered together with this pair of opposite vertices of q conjugate for C_i are enough to make q self-conjugate for C_i.* Hence: *The existence of the three conics q_i is sufficient, while that of two of them is necessary for the self-conjugacy of q with respect to C_i.* I.e. *If two bi-altitudes of a tetrahedron meet the third, the tetrahedron is orthogonal:* Hence: *The concurrency of the three bi-altitudes of a tetrahedron is sufficient, while two of them meeting the third is necessary for the orthogonality of the tetrahedron.*

4. *Other Algebraic Identities.*

a. From (i), (ii) & (iii) we may derive the following three more identities:

$$(g+v)(B) + (h+w)(C) - (f+u)(A) \equiv 2(g+v)(h+w)(A') \dots\dots (viii),$$

$$(h+w)(C) + (f+u)(A) - (g+v)(B) \equiv 2(h+w)(f+u)(B') \dots\dots (ix),$$

$$(f+u)(A) + (g+v)(B) - (h+w)(C) \equiv 2(f+u)(g+v)(C') \dots\dots (x).$$

Eliminating (A′), (B′), (C′) from these three identities with the help of (D′), we have the simplest possible, but not at all obvious, identity, of order 7, that connects (A), (B) & (C), each of order 4, only, from which

*Court, N. A., "The Semi-Orthocentric Tetrahedron, "American Mathematical Monthly, Vol. LX, No. 5, May, 1953, p. 306.

we deduce that two of them are independent, the third follows from the two, i.e. (D)$\equiv\Sigma(f+u)\Sigma(f+u)(A) - 2\Sigma f(f+u)^2(A) \equiv 0$. On interpretation of (D) we find that: *The existence of two of the three conics q_i implies that of the third.* I.e.

If two bi-altitudes of a tetrahedron meet the third, the three CONCUR.

In fact, we could have anticipated this result and should have observed the same earlier, but would have missed (D).

b. We could have gotten (D) from (*v*), (*vi*) & (*vii*) too, of course, with the help of (D′), but we would like to add a few new more to the list, e.g.

$$u\Sigma(f+u)(A) + 2f\Sigma u(A) - 2fu(A) \equiv 2(fvw+ugh)(A') \text{........} (xi),$$

$$v\Sigma(f+u)(A) + 2g\Sigma u(A) - 2gv(B) \equiv 2(gwu+vhf)(B') \text{......} (xii),$$

$$w\Sigma(f+u)(A) + 2h\Sigma u(A) - 2hw(C) \equiv 2(huv+wfg)(C') \text{....} (xiii).$$

The interest of these identities lies in their actual verification, to facilitate the same we may verify and note the two significant as well as simple relations that exist between the values of (A), (B) & (C) as:

$$(1)\ \Sigma u(A) \equiv \Sigma ugh(v^2-w^2);$$

$$(2)\ \Sigma(f+u)(A) \equiv 2\Sigma(g^2-h^2)(fvw-ugh).$$

5. *Observations.*

a. *The three conics q_i are out polar* to, and the quadrilateral q is self-conjugate for, the conic C_i.*

b. The quadrangle q' formed by the four points O, (ll′), (mm′), (nn′) is noticed to be self-conjugate* for C_i , and the three pairs of their complementary joins are seen to meet the sides of the triangle PQR in pairs of conjugate points for C_i. Therefore the pencil of conics through them are out-polar to C_i and meet the sides of the triangle PQR in pairs of conjugate points for C_i, for a pencil of conics meet a line in pairs of points in involution and the pairs of conjugate points for a conic also form an involution. Hence: *The polar triangle, of the diagonal triangle of the quadrilateral q with respect to the conic C_i, together with their centre of perspectivity form a quadrangle q' self-conjugate for C_i such that through q' and each pair of opposite vertices of q there pass a conic out-polar to C_i.*

*Baker, ibid. p. 200-204.

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AN ANALOGUE TO CLIFFORD'S CHAIN

Katsutarô Kobayashi

The purpose of this paper is to prove an analogue to the theorem of Clifford[1] in Euclidean geometry. Let a_i, $i = 1, 2, \dots$, be lines in a plane. Let us write the equation of a_i, referred to complex coordinates[2] x, y, as

$$xt_i + y = x_i t_i .$$

If we take three lines a_1, a_2, a_3 their circumcircle a_{123} is given by

$$x = C_0 - C_1 t$$

where

$$C_0 = \sum \frac{x_1 t_1^{\,2}}{(t_1 - t_2)(t_1 - t_3)}, \quad C_1 = \sum \frac{x_1 t_1}{(t_1 - t_2)(t_1 - t_3)}.$$

Now introduce the point u_{123} by setting

$$u_{123} = C_0 + uC_1, \quad |u| = 1 .$$

Let it be called u-point on a_{123}.

LEMMA 1. If a_3 varies, the locus of u_{123} is a line through the intersection of a_1 and a_2.

PROOF. Let x_{12} be the intersection of a_1 and a_2 and y_{12} its conjugate. Then the locus of u_{123} is given by

$$t_1 t_2 x - uy = t_1 t_2 x_{12} - uy_{12} .$$

LEMMA 2. If we take four lines a_1, a_2, a_3, a_4, four u-points u_{234}, u_{341}, u_{412}, u_{123} lie on a circle through the Wallace point of the four lines.

PROOF. The four u-points are included in

$$(1) \qquad x = \sum \frac{x_1 t_1^{\,2}(t_1 - t)}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)} - \sum \frac{x_1 t_1 (t_1 - t)}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)},$$

since this is u_{123} *when* $t = t_4$, and so on. Now (1) is of the form

$$(2) \qquad x = d_0 - d_1 t .$$

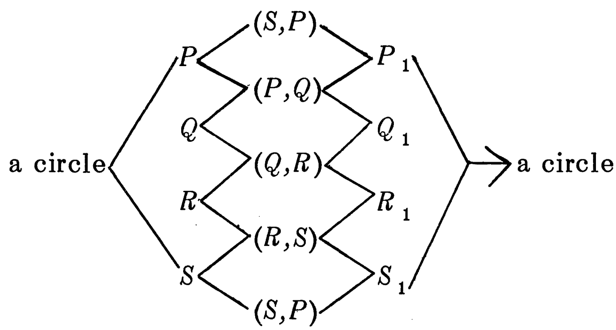
Hence the four u-points lie on a circle, which we denote by c_{1234}. Further,

(2) is the Wallace point when $t = \bar{d}_1 e_1 / d_1 \bar{e}_1$ where

$$e_1 = \sum \frac{x_1 t_1^2}{(t_1-t_2)(t_1-t_3)(t_1-t_4)}.$$

LEMMA 3. If we take four points P, Q, R, S on a circle and draw any circles through S,P; P,Q; Q,R; R,S the remaining intersections of the successive circles are on a circle.

This is equivalent to the theorem of Clifford for four lines, regarding a line as a circle. We express this by the following schema:



where (S,P) is a circle through S,P, and so on, and P_1 is the intersection of (S,P) and (P,Q) different from P, and so on.

Now we shall prove an analogue to the theorem of Clifford in which we denote by $x_{12\ldots n}$ the Clifford's point of n lines $a_1, a_2, \ldots, a_n$ as n is even and by $a_{12\ldots n}$ the Clifford's circle of n lines $a_1, a_2, \ldots, a_n$ as n is odd.

THEOREM. Five lines a_1, a_2, a_3, a_4, a_5 have five circles c_{2345}, c_{3451}, c_{4512}, c_{5123}, c_{1234} which meet in a point u_{12345} on a_{12345}. Six lines $a_1, a_2, \ldots, a_6$ have six such points $u_{23456}, u_{34561}, \ldots, u_{12345}$ which lie on a circle c_{123456} through x_{123456}. And so on.

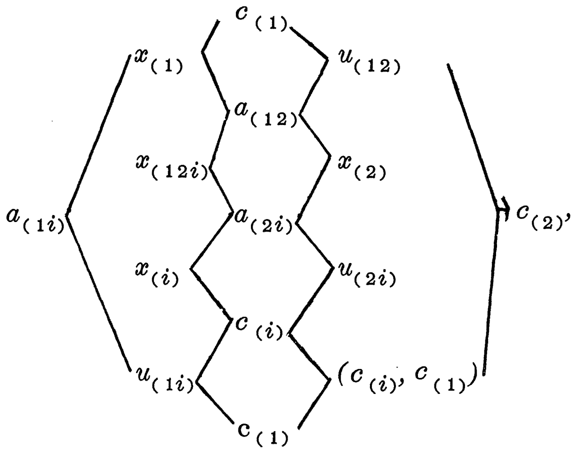
PROOF.[3] We use the following notations:

$$y_{(i)} = y_{12\ldots i-1,\ i+1\ldots n+1},$$

$$y_{(ij)} = y_{12\ldots i-1,\ i+1\ldots j-1,\ j+1\ldots n+1},$$

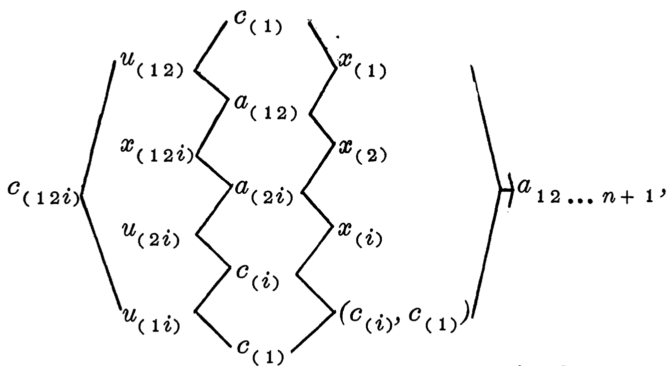
$$y_{(ijk)} = y_{12\ldots i-1,\ i+1\ldots j-1,\ j+1\ldots k-1,\ k+1\ldots n+1}$$

$(n \geq 4)$. Suppose that the sequence of points $u_{12345}, \ldots, u_{12\ldots n}$ and the sequence of circles $c_{123456}, \ldots, c_{12\ldots n-1}$ or the sequence of points $u_{12345}, \ldots, u_{12\ldots n-1}$ and the sequence of circles $c_{123456}, \ldots, c_{12\ldots n}$, as n is odd or even, have already been constructed so as to satisfy the requirements of the theorem. For $n+1$ lines $a_1, a_2, \ldots, a_{n+1}$, if n is even, we have



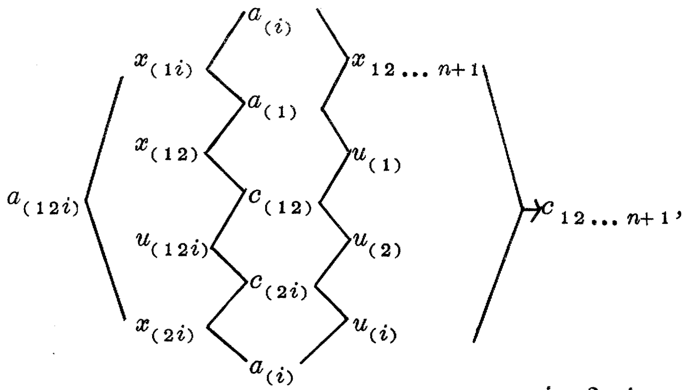
$i = 3, 4, \ldots, n+1,$

where $(c_{(i)}, c_{(1)})$ is the intersection of $c_{(i)}$ and $c_{(1)}$ different from $u_{(1i)}$. Hence the $n+1$ circles $c_{(k)}$, $k = 1, 2, \ldots, n+1$, meet in a point, which we denote by $u_{12\ldots n+1}$. We have also



$i = 3, 4, \ldots, n+1.$

Hence $u_{12\ldots n+1}$ lies on $a_{12\ldots n+1}$. If n is odd, denoting by $c_{12\ldots n+1}$ the circle through $u_{(1)}$, $u_{(2)}$, and $x_{12\ldots n+1}$, we have



$i = 3, 4, \ldots, n+1.$

Hence $x_{12\ldots n+1}$ and the $n+1$ points $u_{(k)}$, $k = 1, 2, \ldots, n+1$, lie on $c_{12\ldots n+1}$. This proves the theorem.

FOOTNOTES

[1] Clifford, Math. Papers (1882).

[2] F. Morley, Trans. Amer. Math. Soc., 1 (1900).

[3] J. H. Grace, Trans. Camb. Phil. Soc., 16 (1898).

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TEST FOR DIVISIBILITY BY THE USE OF A REMAINDER FUNCTION

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In the following paper, an algorithm will be given, by example, for finding the odd factors of an integer N by dividing the successive odd numbers, 3, 5, ... k_n, into an integer N_n, which is smaller than N_{n-1}, and which

(a) Is derived from its predecessor, N_{n-1}, by the recurrent operation which forms the algorithm, and

(b) Divides N without remainder when the remainder function, $r_1 - \sum_{n=2}^{n=n} r_n$, equals zero, or K_n, the last divisor.

An indication of the method of proof, paralleling the steps by which the algorithm was discovered, will then be given.

Example: Test 407 for divisibility.

Step 1. Divide 407 by 3, finding quotient 135, with first remainder, r_1, = 2.

Step 2. Divide twice the first quotient by 5, finding second quotient 54, with remainder, r_2, = 0.

$$2 \cdot 135 = 270$$
$$270 \div 5 = 54;\ r_2 = 0$$
$$r_1 - r_2 = 2 - 0 = 2.$$

Step 3. Subtract twice the second quotient from twice the first quotient, and divide by 7, finding quotient 23, remainder $T_3 = 1$.

$$2 \cdot 135 - 2 \cdot 54 = 162$$
$$162 \div 7 = 23;\ r_3 = 1$$
$$r_1 - r_2 - r_3 = 2 + 0 - 1 = 1.$$

The algorithm is now established, and the recurrent divisions can be continued, until $r_1 - \sum_2^n r_n = 0$, or 11. This occurs for the fifth divisor, $K_5 = 11$. The example, worked out in the form most easily computed, and best calculated to show the recurrent processes, is as follows:

n	K_n	N_n	Q_n	r_n	$r_1 - \sum_2^n r_n$
1	3	407	135	2	2
2	5	270	54	0	2
3	7	162	23	1	1
4	9	116	13	−1	2
5	11	90	8	2	0
6		74			

$$407 = 11 \times 74/2 + 2 - 0 - 1 + 1 - 2 = 11 \times 37.$$

The foregoing algorithm represents a law of number relations, by which any integral number, N, may be tested. Some of its characteristics and principles of usage, are as follows:

(a) The test applies to both even and odd numbers; but, with an even number, divide by 2 until an odd number is reached. Then test the latter number by the algorithm.

(b) Note that the divisors taken are the ascending odd numbers, 3, 5, 7, ... in succession.

(c) The quotients, found by short or long arithmetic division, are selected so as to give $0 \leqq r_1 - \sum_2^n < K_n$. At the nth step, Q_n may be selected so that either $r_1 - \sum_2^n r_n = 0$ or $r_1 - \sum_2^n r_n = K_n$. This happens only when K_n is a divisor of N. When $r_1 - \sum_2^n r_n = 0$, $\frac{N_{n+1}}{2}$ = the other factor of N.

(d) The remainders are given their normal + or − signs, depending on whether the divisor × the quotient is less than, or greater than N_n. In the summation of remainders, constituting $\Phi(r)$, the first remainder retains its normal sign. The other remainders, after the first, change signs.

$$\Phi(r) = r_1 - r_2 - r_3 \ldots, - r_n.$$

(e) If K_n, the nth divisor, passes the integer for which the square is closest to N, and less than N (i.e., $K_n = [\sqrt{N}]$), without $\Phi(r)$ equalling 0 or K_n, N is a prime number, and the first divisor, after unity, is N itself.

Indication of Method of Development of the Algorithm and Its Proof:

A number, N, is expressible in a number base, by taking successive divisions and remainders, and using the remainders as the coefficients of the successive powers of the divisor. Thus, 4271 to the base ten is developed by the algorithm:

	N_n	Q_n	r_n
10)	4271	427	1
10)	427	42	7
10)	42	4	2
	4		

$$4271 = 4 \times 10^3 + 2 \times 10^2 + 7 \times 10 + 1 = (4,2,7,1)_{10}$$

The number N, to the base 10, may be converted to another base, by the same kind of algorithm. Thus, 4271, to the base 7, is derived as follows:

7	4271	610	1
7	610	87	1
7	87	12	3
7	12	1	5
	1		

$$(4,2,7,1)_{10} = 1\times 7^4 + 5\times 7^3 + 3\times 7^2 + 1\times 7 + 1 = (1,5,3,1,1)_7$$

The number base need not be a fixed integer. Factorial, or other sequences may be used. Thus:

2)	4271	2135	1
3)	2135	711	2
4)	711	177	3
5)	177	35	2
6)	35	5	5
	5		

$$(4,2,7,1)_{10} = 5\underline{)6} + 5\underline{)5} + 2\underline{)4} + 3\underline{)3} + 2\underline{)2} + 1 = (5,5,2,3,2,1)_{\text{factorial base}}.$$

In place of integers, we may substitute fractions as divisors. Thus:

2)	91	45	1
3/2)	45	30	0
4/3)	30	22	2/3
5/4)	22	17	3/4
6/5)	17	14	1/5
7/6)	14	12	0
	12		

$$\begin{aligned}91 = {} & 12(7/6 \cdot 6/5 \cdots 3/2 \cdot 2) \\ & + 0(6/5 \cdot 5/4 \cdots 3/2 \cdot 2) \\ & + 1/5(5/4 \cdot 4/3 \cdot 3/2 \cdot 2) \\ & + 3/4(4/3 \cdot 3/2 \cdot 2) \\ & + 2/3(3/2 \cdot 2) \\ & + 0(2) \\ & + 1\end{aligned}$$

$$91 = 7\times 12 + 1 + 3 + 2 + 1 = 84 + 7.$$

By suppressing the denominators of the fractions, we arrive at an algorithm equivalent to the preceding algorithm, but with changed signs for remainders after the first:

Thus:

2)	91	45	1
3)	45	15	0
4)	30	8	−2
5)	22	5	−3
6)	17	3	−1
7)	14	2	0
	12		

$$91 = 7 \times 12 + 1 + 3 + 2 + 1.$$

Fractions with even numerators may be suppressed.
Thus:

3)	407	135	2
5/3)	135	81	0
7/5)	81	58	−1/5
9/7)	58	45	+1/7
11/9)	45		−2/9
	37		

$$\begin{aligned} 407 &= 37(11/9 \cdot 9/7 \cdots 3) \\ &\quad -2/9(9/7 \cdot 7/5 \cdots 3) \\ &\quad +1/7\,(7/5 \cdots 3) \\ &\quad -1/5\,(5/3 \cdot 3) + 0 + 2 \\ 407 &= 11 \times 37 - 2 + 1 - 1 + 2 = 11 \times 37 \end{aligned}$$

And, converting the foregoing algorithm with fractions to an all-integer algorithm, we have:

				$r_1 - \sum_2^n r_n$
3)	407	135	2	2
5)	270	54	0	2
7)	162	23	1	1
9)	116	13	−1	2
11)	90	8	+2	0
	74			

$$407 = 11 \times 74/2 + 2 - 1 + 1 - 2 = 11 \times 37.$$

But this is the subject algorithm.

The logical connections and recurrences involved are illustrated by the foregoing examples, which, by the substitution of appropriate symbols for numbers, may readily be generalized. A large number of interesting number relationships, all depending on the subject algorithm, may then be developed. By suitably modifying the algorithm, composite odd divisors may be suppressed, and only successive primes used as divisors. However, as indicated in the opening paragraph, it is desired, in this paper, to go no further than to illustrate the algorithm by means of examples, to suggest one method of proof, and show how the algorithm was arrived at.

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ON THE POLAR PROJECTION WITH RESPECT TO NORMAL CURVES

Masaru Inagaki

SYNOPSIS

Draw an enveloping hypercone of a fixed hypersphere S from a point P in an n-dimensional projective space P_n and project their points of contact on a fixed hyperplane P_{n-1} of P_n from a point O of S, then we get a hypersphere K in P_{n-1}. This projection $P \to K$ from P_n to P_{n-1} is the so called polar projection with respect to the hypersphere S and plays an important role for discussing the relation between the conformal space and the projective space. As S is a sort of quadratic hypersurfaces, it may be regarded as a highest dimensional algebraic subvariety of the lowest order in P_n. It will be interesting to study about an analogous projection replacing the hypersphere by a normal curve, that is to say, the curve whose lowest dimensional ambient space is of dimensionality n. The main objects of this paper are to study some fundamental properties of this projection and some of its applications.

1. DEFINITION

An algebraic curve C_n of the n-th order whose lowest dimensional ambient space is P_n is usually said to be a normal curve and is given by the following equations:

$$\rho x_i = t^{n-i+1} \quad (i = 1, \ldots, n+1, \ t\text{: parameter}) \tag{1.1}$$

Describe an osculating hyperplane $\sum_{i=1}^{n+1} a_i x_i = 0$ to the normal curve through a point $P(p_1, \ldots, p_{n+1})$ of P_n, then the coordinates of the point of contact to the curve must satisfy the following equations

$$\sum_{i=1}^{n+1} a_i p_i = 0, \qquad \sum_{i=1}^{n+1} a_i t^{n-i+1} = 0. \tag{1.2}$$

At this point of contact the curve and the hyperplane have n points in common. The equation $(1.2)_2$ must have n-ple root. Therefore we can write

$$\sum_{i=1}^{n+1} a_i t^{n-i+1} \equiv a_1 (t + \alpha)^n$$

where $-\alpha$ is the n-ple root.

Comparing the coefficients of each side of the last equation term by term, we have

(1.3) $a_2/a_1 = \binom{n}{1}\alpha,\ a_3/a_1 = \binom{n}{2}\alpha^2,\ \dots,\ a_{n+1}/a_1 = \alpha^n.$

Introducing these into $(1.2)_1$, we get the following equation of the n-th order concerning α:

$$\sum_{i=1}^{n+1} p_i\binom{n}{i-1}\alpha^{i-1} = 0. \tag{1.4}$$

Since each root of this equation determines a set of ratios

$$a_1 : a_2 : \dots : a_{n+1},$$

we can describe n osculating hyperplanes of the normal curve through the point P.

Now, fix a point O_n on C_n. When a point P is given in P_n, and if we project from O_n the points of contact of the ocsulating hyperplanes to K_n drawn through P on a fixed $(n-1)$-dimensional linear subspace P_{n-1} of P_n, we get n points on a fixed normal curve C_{n-1} of P_{n-1} which is the projection of C_n from O_n on P_{n-1}. Then, projecting these n points from a point O_{n-1} of C_{n-1} on a fixed $(n-2)$-dimensional linear subspace P_{n-2} of P_{n-1}, we get again n points on a fixed normal curve C_{n-2} of P_{n-2} which is the projection of C_{n-1} from O_{n-1} on P_{n-2}. Repeating the same process we get n points on a fixed normal curve C_m of P_m where P_m is a linear subspace of P_{m+1} $(m = n-1, n-2, \dots)$.

In such a way we get a one to one correspondence between points of P_n and a set of n points on a fixed normal curve of P_m. We call this correspondence the polar projection with respect to the normal curves C_j $(j = n, n-1, \dots, m)$.

2. *FUNDAMENTAL PROPERTIES*

THEOREM (2.1) The hyperplane determined by the n points of contact of the osculating hyperplanes to a normal curve C_n of P_n drawn from any point of P_m in P_n passes through the same $(n-m-1)$-dimensional linear subspace of P_n.

PROOF. Take $m+1$ linearly independent points

$$(a_i^{(1)}),\ (a_i^{(2)}),\ \dots,\ (a_i^{(m+1)}) \qquad (i = 1, \dots, n+1) \tag{2.1}$$

in P_n, then the points of the m-dimensional subspace P_m spanned by these points are

$$\rho y_i = \sum_{k=1}^{m+1} \lambda_k a_i^{(k)} \qquad (i = 1, \dots, n+1) \tag{2.2}$$

where λ_k's are parameters. Let an osculating hyperplane to C_n drawn from the point (y_i) be $\sum_{i=1}^{n+1} p_i x_i = 0$, then the parameter of C_n at the point of contact is a root of the following equation

$$\sum_{i=1}^{n+1} p_i t^{n-i+1} = 0. \tag{2.3}$$

As the roots of the preceding equation are n-ple, we have

$$\sum_{i=1}^{n+1} p_i t^{n-i+1} \equiv p_1 (t+\alpha)^n,$$

hence we get

$$p_i/p_1 = \binom{n}{i-1} \alpha^{i-1}. \tag{2.4}$$

Now, as y_i's satisfy the equation $\sum_{i=1}^{n+1} p_i x_i = 0$, we have

$$\sum_{i=1}^{n+1} \sum_{k=1}^{m+1} \binom{n}{i-1} \alpha^{i-1} \lambda_k a_i^{(k)} = 0. \tag{2.5}$$

It is clear that its roots α_l $(l = 1, \ldots, n)$ are the parameters of C_n at the points of contact of osculating hyperplanes. Therefore the hyperplane determined by these points of contact is given by the following equation

$$\begin{vmatrix} x_1 & x_2 & \cdots\cdots & x_{n+1} \\ \alpha_1^n & \alpha_1^{n-1} & \cdots\cdots & 1 \\ & & \cdots\cdots\cdots & \\ \alpha_n^n & \alpha_n^{n-1} & \cdots\cdots & 1 \end{vmatrix} = 0. \tag{2.6}$$

Multiply the i-th column of the last determinant by $\sum_{k=1}^{m+1} \binom{n}{n-i+1} \lambda_k a_{n-i+1}^{(k)}$ for every i $(i = 1, \ldots, n+1)$ and add them to the first column. We can see immediately from (2.5) that all elements of the first column except (1,1) element vanish. So we can write the equation of our hyperplane as follows:

$$\sum_{k=1}^{m+1} \lambda_k \left(\sum_{i=1}^{n+1} \binom{n}{n-i+1} a_{n-i+1}^{(k)} x_i \right) = 0. \tag{2.7}$$

The last equation is evidently the linear combination of $m+1$ hyperplanes

$$\sum_{i=1}^{n+1} \binom{n}{n-i+1} a_{n-i+1}^{(k)} x_i = 0, \tag{2.8}$$

so the hyperplane (2.7) passes through an $(n-m-1)$-dimensional linear subspace which is the intersection of the hyperplanes (2.8).

THEOREM (2.2). By the polar projection $P_n \to P_{m+1}$ any point P of P_n corresponds to a set of n points on a fixed normal curve in P_{m+1}. If P moves on a fixed m-dimensional linear subspace P_m^0 of P_n, then the vertices of the complete n-hedron determined by the osculating hyperplanes of C_{m+1} drawn at each point of the n points corresponding to P lie on a fixed m-dimensional algebraic variety of the $(n-m)$-th order. (By the complete n-hedron we mean the analogue of the complete quadrilateral in P_2).

PROOF. From Theorem (2.1), we can see that the hyperplane determined by the points of contact of the osculating hyperplanes to C_n drawn from any point P of P_m^0 is given by the following equation

$$(2.9) \qquad \sum_{i=1}^{n+1} \sum_{k=1}^{m+1} \lambda_k a_i^{(k)} x_i = 0.$$

Therefore the parameters of C_n which give the points of contact of the osculating hyperplanes through P are the roots $t_1, \ldots, t_n$ of the equation

$$(2.10) \qquad \sum_{i=1}^{n+1} \sum_{k=1}^{m+1} \lambda_k a_i^{(k)} t^{n-i+1} = 0,$$

this being an immediate consequence of (1.1) and (2.9). Project these points from the point $(\overbrace{1, 0, \ldots, 0}^{n+1})$ into the $(n-1)$-dimensional linear subspace $P_{n-1} : x_1 = 0$, then we obtain a set of n points

$$(2.11) \qquad (t_l^{n-1}, \ldots, t_l, l) \qquad (l = 1, \ldots, n)$$

lying on C_{n-1}. Project these points from the point $(\overbrace{1, 0, \ldots, 0}^{n})$ of P_{n-1} into the $(n-2)$-dimensional linear subspace $P_{n-2} : x_1 = 0,\ x_2 = 0$. Then we obtain n points

$$(2.12) \qquad (t_l^{n-2}, \ldots, t_l, l) \qquad (l = 1, \ldots, n)$$

lying on C_{n-2}. Proceeding in this way we obtain ultimately n points

$$(2.13) \qquad (t_l^{m+1}, \ldots, t_l, l) \qquad (l = 1, \ldots, n)$$

lying on C_{m+1}. And the equations of the osculating m-planes of the normal curve C_{m+1} at these points are given, after changing the unit point of the $(m+1)$-simplex of reference in P_{m+1} suitably, by the following equations

(2.14) $t_l^{m+1}x_{n-m} + \dots + t_l x_n + x_{n+1} = 0 \qquad (l = 1, \dots, n).$

Now, let the point of intersection of $m+1$ osculating m-planes of C_{m+1} at the points where the parameters are $t_{s_1}, \dots, t_{s_{m+1}}$ $(s_1, \dots, s_{m+1} = 1, \dots, n)$ respectively be $(\xi_{n-m}, \dots, \xi_n, \xi_{n+1})$. Then we have

(2.15) $$t^{m+1}\xi_{n-m} + t^m\xi_{n-m+1} + \dots + t\xi_n + \xi_{n+1} = \xi_{n-m}\prod_{k=1}^{m+1}(t-t_{s_k}).$$

As the roots of the equation (2.10) are $t_1, \dots, t_n$, we have

(2.16) $$\sum_{i=1}^{n+1}\sum_{k=1}^{m+1}\lambda_k a_i^{(k)}t^{n-i+1} = \sum_{k=1}^{m+1}\lambda_k a_1^{(k)}\prod_{l=1}^{n}(t-t_l),$$

so the left side of (2.16) can be divided by that of (2.15). Therefore the left side of (2.16) is a linear combination of the terms obtained by the left side of (2.15) multiplied by t^λ $(\lambda = 0, 1, \dots, n-m-1)$.

Consequently it follows that the point $(\xi_{n-m}, \dots, \xi_{n+1})$ satisfies the following equation

(2.17) $$\begin{vmatrix} a_1^{(1)} & a_2^{(1)} & \dots\dots\dots & a_{n+1}^{(1)} \\ & \dots\dots\dots & & \\ a_1^{(m+1)} & a_2^{(m+1)} & \dots\dots\dots & a_{n+1}^{(m+1)} \\ x_{n-m} & x_{n-m+1} \dots x_{n+1} & 0 \dots\dots & 0 \\ & \dots\dots\dots & & \\ 0 & 0 \dots 0 & x_{n-m} \dots\dots & x_{n+1} \end{vmatrix} = 0.$$

Hereafter we shall denote this equation in the following form

(2.18) $$[\mathrm{A}^{(1)}\mathrm{A}^{(2)} \dots \mathrm{A}^{(m+1)}\mathrm{X}_{(1)} \dots \mathrm{X}_{(n-m)}] = 0$$

where $\mathrm{A}^{(k)}$'s and $\mathrm{X}_{(j)}$'s are $(1, n+1)$ matrices defined as follows

$$\mathrm{A}^{(k)} \equiv (a_1^{(k)} \dots a_{n+1}^{(k)}), \qquad (k = 1, \dots, m+1)$$
$$\mathrm{X}_{(1)} \equiv (x_{n-m} \dots x_{n+1}\ 0 \dots 0)$$
$$\mathrm{X}_{(2)} \equiv (0\ x_{n-m} \dots x_{n+1}\ 0 \dots 0)$$
$$\dots\dots\dots$$
$$\mathrm{X}_{(n-m)} \equiv (0 \dots 0\ x_{n-m} \dots x_{n+1})$$

and [ ] means a determinant having the matrices between the square bracket as its rows.

(2.18) is a fixed m-dimensional variety of the $(n-m)$-th order. Therefore the point of intersection of every $m+1$ set of osculating m-planes of (2.14) lies on this variety, as was to be proved.

THEOREM (2.3). If an m-dimensional linear subspace P_m^0 of P_n is contained in a k-dimensional linear subspace P_k^0 of P_n, then the m-dimensional algebraic variety of the $(n-m)$-th order obtained by the polar projection of P_m^0 into an $(m+1)$-dimensional linear space P_{m+1} passes through a fixed $(2m-k)$-dimensional algebraic variety of the $\{(n-k)^{k-m+1} - (n-k-1)^2\}$-th order provided that $m < k \leqq 2m$. The converse is also true.

PROOF. At first we assume that $k = m+1$.

Take $m+2$ mutually independent points $L_1, \ldots, L_{m+2}$ on P_k^0 so that $m+1$ points $L_1, \ldots, L_{m+1}$ are independent points in P_m^0 too. Then the m-dimensional algebraic variety of the $(n-m)$-th order obtained by the polar projection of P_m^0 into an $(m+1)$-dimensional space P_{m+1} has the following equation:

$$[\mathrm{A}^{(1)} \ldots \mathrm{A}^{(m+1)} \mathrm{X}_{(1)} \ldots \mathrm{X}_{(n-m)}] = 0. \tag{2.20}$$

Now, consider the set of all points on (2.20) such that the rank r of a matrix

$$(\mathrm{A}^{(1)} \ldots \mathrm{A}^{(m+2)} \mathrm{X}_{(1)} \ldots \mathrm{X}_{(n-m)})$$

is n; it contains the intersection of the following set of m-dimensional varieties of order $n-m-1$

$$[\mathrm{A}^{(1)} \ldots \mathrm{A}^{(m+2)} \mathrm{X}_{(1)} \ldots \mathrm{X}_{(j-1)} \mathrm{X}_{(j+1)} \ldots \mathrm{X}_{(n-m)}] = 0 \tag{2.21}$$
$$(j = 1, \ldots, n-m, \ \mathrm{X}_{(0)} \equiv \mathrm{X}_{(n-m)}, \ \mathrm{X}_{(n-m+1)} \equiv \mathrm{X}_{(1)}).$$

Because, if we assume formally that $\mathrm{A}^{(k)}$ and $\mathrm{X}_{(j)}$ be the coordinates of points in an n-dimensional projective space, then (2.21) shows that, when $r = n$, the points $\mathrm{A}^{(1)}, \ldots, \mathrm{A}^{(m+2)}, \mathrm{X}_{(1)} \ldots, \mathrm{X}_{(n-m)}$ belong to a same hyperplane of this space. Of course the points $\mathrm{A}^{(1)}, \ldots, \mathrm{A}^{(m+1)}, \mathrm{X}_{(1)}, \ldots, \mathrm{X}_{(n-m)}$ lie on this hyperplane, therefore the equation (2.20) is satisfied. Hence the set of points on the algebraic variety of (2.20) such that $r = n$ contains the intersection of the varieties of (2.21).

Among the equations of (2.21) only two are independent, as is easily seen. Therefore the intersection of the varieties of (2.21) is of dimension $m-1$.

Now, consider the intersection of two m-varieties

$$\begin{aligned} &[\mathrm{A}^{(1)} \ldots \mathrm{A}^{(m+2)} \mathrm{X}_{(1)} \mathrm{X}_{(3)} \ldots \mathrm{X}_{(n-m)}] = 0 \\ &[\mathrm{A}^{(1)} \ldots \mathrm{A}^{(m+2)} \mathrm{X}_{(2)} \mathrm{X}_{(3)} \ldots \mathrm{X}_{(n-m)}] = 0. \end{aligned} \tag{2.22}$$

It is clear that the set of points in this intersection such that $r=n$ coincides with the intersection of (2.21). In the set of points of the intersection of (2.22) such that $r<n$, all minor determinants of the n-th order of the matrix

$$(A^{(1)} \dots A^{(m+2)} X_{(1)} \dots X_{(n-m)})$$

are equal to zero. Therefore if we denote the $(1,n)$ matrices taken off the i-th elements from $A^{(k)}$ and $X_{(k)}$ by $A_i^{(k)}$ and $X_{(k)i}$ respectively, the last set of points lie on the intersection of the following varieties of the $(n-m-2)$-th order

$$(2.23) \qquad [A_i^{(1)} \dots A_i^{(m+2)} X_{(3)i} \dots X_{(n-m)i}] = 0 \quad (i = 1, \dots, n+1).$$

The intersection of these varieties is not contained in the following varieties

$$(2.24) \qquad [A^{(1)} \dots A^{(m+2)} X_{(1)} \dots X_{(i-1)} X_{(i+1)} \dots X_{(n-m)}] = 0 \quad (i = 3, \dots, n-m).$$

We see therefore that the intersection of (2.22) breaks up into two parts: one is the intersection of (2.23) and its order is $(n-m-2)^2$, the other is the intersection of the varieties of (2.24) and its order is $(n-m-1)^2-(n-m-2)^2$. Hence the order of the intersection of (2.21) must be $(n-m-1)^2-(n-m-2)^2$. Therefore the m-dimensional algebraic varieties obtained by the polar projection of points of P_m^0 into an $(m+1)$-dimensional space pass through the same $(m-1)$-dimensional algebraic variety of the $\{(n-m-1)^2-(n-m-2)^2\}$-th order.

Now, take any other points $Q_1, \dots, Q_{m+2}$ on P_k^0 and construct the varieties analogous to (2.22) and (2.23) about these points. It may be easily seen that the varieties thus obtained are just the same as the original ones, because Q_i's are linear combinations of L_i's. Therefore we see that the result obtained here is valid for any m-dimensional linear subspace of P_k^0. So the first part of our theorem is proved for the case $k=m+1$.

In the next place, we assume that $k = m+2$.

Take $m+3$ mutually independent points $L_1, \dots, L_{m+3}$ on P_k^0 so that the $m+1$ points $L_1, \dots, L_{m+1}$ are independent points in P_m^0 too. Then the algebraic variety obtained by the polar projection of P_m^0 into an $(m+1)$-dimensional space is given by (2.20). Let an $(m+1)$-dimensional space spanned by $L_1, \dots, L_{m+2}$ be P_{m+1}^0.

Now consider the set of all points on the intersection of (2.21) corresponding to P^0_{m+1} by the polar projection such that the rank r' of the matrix

$$(A^{(1)} \dots A^{(m+3)} X_{(1)} \dots X_{(n-m)})$$

is equal to or greater than n; it is shown as before that this set of points contains the intersection of the following set of varieties of order $n-m-2$

$$(2.25) \quad \begin{array}{c} [A^{(1)} \dots A^{(m+3)} X_{(1)} \dots X_{(i-1)} X_{(i+1)} \dots X_{(j-1)} X_{(j+1)} \dots X_{(n-m)}] = 0 \\ (i,j = 1, \dots n-m, \ i \neq j, \ X_{(0)} \equiv X_{(n-m)}, \ X_{(n-m+1)} \equiv X_{(1)}). \end{array}$$

As the variety (2.20) contains the intersection of varieties of (2.21), it also contains the intersection of the varieties of (2.25). Among the equations of (2.25) only three are independent. Therefore the intersection of the varieties of (2.25) is of dimension $m-2$.

Now, consider the intersection of three varieties

$$(2.26) \quad \begin{array}{l} [A^{(1)} \dots A^{(m+3)} X_{(1)} X_{(4)} \dots X_{(n-m)}] = 0 \\ [A^{(1)} \dots A^{(m+3)} X_{(2)} X_{(4)} \dots X_{(n-m)}] = 0 \\ [A^{(1)} \dots A^{(m+3)} X_{(3)} X_{(4)} \dots X_{(n-m)}] = 0. \end{array}$$

In the set of points such that $r' < n$, all minor determinants of the n-th order of

$$(A^{(1)} \dots A^{(m+3)} X_{(1)} \dots X_{(n-m)})$$

are equal to zero; this set lies on the intersection of the following varieties of the $(n-m-3)$-th order

$$(2.27) \quad \begin{array}{c} [A^{(1)}_j \dots A^{(m+3)}_j X_{(4)j} \dots X_{(n-m)j}] = 0 \\ (j = 1, \dots, n+1). \end{array}$$

The intersection of these varieties is not contained in the following varieties

$$(2.28) \quad \begin{array}{c} [A^{(1)} \dots A^{(m+3)} X_{(1)} \dots X_{(i-1)} X_{(i+1)} \dots X_{(j-1)} X_{(j+1)} \dots X_{(n-m)}] = 0. \\ (i,j = 4, \dots, n-m) \end{array}$$

We see therefore that the intersection of (2.26) breaks up into two parts: one is the intersection of (2.27) and its order is $(n-m-3)^2$, the other is the intersection of the varieties (2.28) i.e. the intersection of (2.25) and its order is $(n-m-2)^3 - (n-m-3)^2$. Therefore the n-dimensional algebraic varieties obtained by the polar projection of points of P^0_m in P^0_{m+2} into an $(m+1)$-dimensional space contains the same $(m-2)$-dimensional algebraic variety of the $(n-m-2)^3-(n-m-3)^2$-th order.

The equations of (2.25) are not dependent on the choice of the fundamental points L_i in P_k^0. Therefore we see that the result obtained here is valid for any m-dimensional subspace of P_k^0. So the first part of our theorem is proved for the case $k=m+2$.

By the method of mathematical induction, we may deduce the first part of our theorem for the general case.

The converse of the first part is verified easily.

COROLLARY. If a k-dimensional subspace P_k^0 of P_n passes through a fixed m-dimensional subspace P_m^0 of P_n, then the $(2n-k)$-dimensional algebraic variety of the $\{(n-k)^{k-m+1}-(n-k-1)^2\}$-th order corresponding to P_k obtained by the polar projection of the points of P_n into an $(m+1)$-dimensional space lies on a fixed m-dimensional algebraic variety of the $(n-m)$-th order. The converse is also true.

THEOREM (2.4). By the polar projection of any point of an m-dimensional linear subspace P_m^0 of P_n into an h-dimensional linear space P_h we get a set of n points of a fixed normal curve C_h of P_h. The vertices of the complete n-hedron obtained by the osculating hyperplanes of the normal curve drawn at each point of the set of n points lie on a fixed m-dimensional algebraic variety which is the intersection of $(h-m)$ varieties of the $(n-h+1)$-th order having an $(h-2)$-dimensional variety of the $\{(n-h)^2-(n-h-1)^2\}$-th order in common.

PROOF. Take an h-dimensional subspace P_h^0 of P_n such that P_m^0 is in P_h^0. Let L_i $(i=1,\dots,h+1)$ be $h+1$ independent points of P_h^0 such that the first $m+1$ points are independent points of P_m^0. Then we may regard P_m^0 as the intersection of $h-m$ linear spaces of dimension $(h-1)$ spanned by L_i $(i=1,\dots,m+1)$ and $h-m-1$ other points of L_j $(j=m+2,\dots,h+1)$. The polar projection of these spaces into an h-dimensional space P_h gives rise to the following $h-m$ algebraic varieties

$$[A^{(1)}\dots A^{(m+1)}A^{(m+2)}\dots A^{(i-1)}A^{(i+1)}\dots A^{(h+1)}X_{(1)}\dots X_{(n-m)}]=0$$
$$(2.29)\qquad (i=m+2,\dots,h+1,\ A^{(h+2)}\equiv A^{(m+2)}),$$

where the variables in $X_{(i)}$'s are $(x_{n-h+1},\dots,x_{n+1})$.

These spaces are subspaces of P_h^0. Therefore by the preceding theorem, the varieties of (2.29) have an $(h-2)$-dimensional variety of the $\{(n-h)^2-(n-h-1)^2\}$-th order in common. This intersection is also contained in the variety

$$[A^{(2)}\dots A^{(h+1)}X_{(1)}\dots X_{(n-h+1)}]=0.$$

which is obtained by the polar projection of an $(h-1)$-dimensional subspace spanned by $L_2, \ldots, L_{h+1}$. Therefore it does not contain P_m^0. So the intersection of the varieties of (2.29) is separated into two varieties, the one is the $(h-2)$-dimensional variety mentioned above and the other is the figure corresponding to P_m^0. Let us denote this second variety by

$$(2.30) \qquad [A^{(1)}\ldots A^{(m+1)}(A^{(m+2)}\ldots A^{(h+1)})X_{(1)}\ldots X_{(n-h+1)}] = 0.$$

We see therefore that the figure obtained by the polar projection of P_m^0 into P_h is given by (2.30).

Now let us prove the following theorem which is a generalization of Theorem (2.3).

THEOREM (2.5). If an m-dimensional subspace P_m^0 of P_n is contained in a fixed k-dimensional linear subspace P_k^0 of P_n, then the m-dimensional algebraic variety obtained by the polar projection of P_m^0 into an h-dimensional linear space P_h (c.f. Theorem (2.4)) passes through a fixed $(m+h-k-1)$-dimensional subvariety of P_h provided that $h-1 < k \leqq m+h-1$.

PROOF. At first we assume that $k = m+2$.

Let a basis of P_k^0 be L_i $(i = 1, \ldots, m+3)$. Then, following to the preceding theorem, the polar projection of the $(m-t)$-dimensional space P_{m-t}^0 spanned by the points $L_1, \ldots, L_{m-t+1}$ determines an $(m-t)$-dimensional variety in an $(m+1)$-dimensional space. The equation of the last variety is given by any one of the following

$$(2.31) \qquad \begin{gathered}[A^{(1)}\ldots A^{(m-t+1)}(A^{(m-t+2)}\ldots A^{(i-1)}A^{(i+1)}\ldots A^{(m+3)})X_{(1)}\ldots X_{(n-m)}] = 0 \\ (i = m-t+2, \ldots, m+3, \ A^{(m+4)} \equiv A^{(m-t+2)}).\end{gathered}$$

Now, owing to Theorem (2.3), the $(m+1)$-dimensional spaces spanned by $m+2$ points $L_1, \ldots, L_{i-1}, L_{i+1}, \ldots, L_{m+3}$ $(i = 1, \ldots, m+3)$ determine $(m-i)$-dimensional varieties given as the intersection of the following varieties:

$$(2.32) \qquad \begin{gathered}[A^{(1)}\ldots A^{(i-1)}A^{(i+1)}\ldots A^{(m+3)}X_{(1)}\ldots X_{(j-1)}X_{(j+1)}\ldots X_{(n-m)}] = 0 \\ (j = 1, \ldots, n-m, \ X_{(0)} \equiv X_{(n-m)}, \ X_{(n-m+1)} \equiv X_{(1)}).\end{gathered}$$

Among these there are $t+2$ varieties which are obtained by the polar projection of the $(m+1)$-dimensional spaces passing through the points $L_1, \ldots, L_{m-t+1}$. It is evident that the $(m-t-2)$-dimensional intersection of these $t+2$ varieties is contained in the variety (2.31).

If the $(m-t)$-dimensional space changes its position in P_k^0, the preceding relation still holds good. Therefore, if an $(m-t)$-dimensional space is contained in a fixed k-dimensional space (here $k=m+2$), the $(m-t+1)$-dimensional variety obtained by the polar projection of the points of P^0_{m-t} into an $(m+1)$-dimensional space passes through a fixed $(m-t-2)$-dimensional variety.

In the next place we assume that $k=m+3$.

Let a basis of P^0_k be L_i $(i = 1, \ldots, m+4)$, and determine $m+4$ linear subspaces of dimension $m+2$ spanned by every set of $m+3$ points of L_i's. Applying the polar projection, we obtain the following $m+4$ varieties of dimension $m-2$ corresponding to these $(m+2)$-dimensional subspaces (c.f. Theorem (2.3)).

$$[A^{(1)} \ldots A^{(i-1)} A^{(i+1)} \ldots A^{(m+4)} X_{(1)} \ldots X_{(j-1)} X_{(j+1)} \ldots X_{(l-1)} X_{(l+1)} \ldots X_{(n-m)}]=0 \tag{2.33}$$
$$(i = 1, \ldots, m+4, \ j, l = 1, \ldots, n-m, \ j \neq l).$$

Among these there are $t+3$ spaces which are obtained by the polar projection of the $(m+2)$-dimensional spaces passing through the points L_1, $\ldots$, L_{m-t+1}. It is easily shown that the $(m-t-3)$-dimensional intersection of these $t+3$ varieties is contained in the spaces (2.33).

If the subspace spanned by the points $L_1, \ldots, L_{n-t+1}$ changes its position in P^0_k, the preceding relation still holds good. Therefore if an $(m-t)$-dimensional space be a subspace of a fixed k-dimensional space (here $k=m+3$), then the $(m-t+1)$-dimensional variety obtained by the polar projection of P^0_{m-t} into an $(m+1)$-dimensional space passes through a fixed $(m-t-3)$-dimensional variety.

Repeating similar discussion, and replacing $m-t$ by m and $m+1$ by h, we obtain finally the result mentioned in the theorem.

3. SOME APPLICATIONS

Owing to the fundamental properties shown in the preceding paragraph, we may transform certain linear subspaces into subvarieties of any dimension which are intimately related to normal curves. Making use of this idea we can deduce some properties concerning normal curves by transforming simple properties of linear spaces. Here we shall explain some of them.

Applying Theorem (2.2) to the proposition "$m+1$ independent points span an m-dimensional projective space" we have the following

THEOREM (3.1) The vertices of n complete m-hedrons osculating to an n-dimensional normal curve lie on a variety of the $(m-n+1)$-th order. For, by virtue of Theorem (2.2), $m+1$ points are transformed into the $m+1$ n-hedrons in an $(m+1)$-dimensional space and their vertices must lie on a hypersurface of the $(n-m)$-th order. Replacing m by $n-1$ and n by m, the theorem follows immediately.

As a special case of Theorem (3.1), we have the following

COROLLARY 1. The vertices of two complete m-latera circumscribed to a conic lie on a curve of the $(m-1)$-th order.

This is an immediate generalization of Brianchon's theorem.

COROLLARY 2. The vertices of three osculating complete m-hedrons of a twisted cubic lie on a surface of the $(m-2)$-th order.

The case when $m=4$ is well known, and the case when $m=5$ is due to Pasch.

Theorem (3.1) may be generalized in the following form:

THEOREM (3.2). The vertices of k complete m-hedrons osculating to an n-dimensional normal curve lie on the intersection of $n-k+1$ varieties of order $m-n+1$ having an $(n-2)$-dimensional subvariety of the $\{(m-n)^2-(m-n-1)^2\}$-th order in common.

This follows from Theorem (2.4). The case when $k=2$, $m=4$ and $n=3$ is known as follows:

The vertices of two osculating tetrahedrons of a twisted cubic lie on another twisted cubic.

If we put $k=2$ and $m=3$ in Theorem (3.2), we get a generalized theorem of the preceding one, that is:

COROLLARY 3. The vertices of two osculating complete m-hedrons of a twisted cubic lie on a curve of order $(m-2)^2-(m-3)^2+(m-4)^2$.

From Theorem (2.3), we easily obtain the following

THEOREM (3.3). The varieties of order $m-n+1$ (c.f. Theorem (3.1)) passing through each set of all vertices of n osculating m-hedrons among $n+1$ osculating m-hedrons of an n-dimensional normal curve pass through a fixed $(n-2)$-dimensional algebraic variety of the $\{(m-n)^2-(m-n-1)^2\}$-th order. Each of the preceding variety determined by each set of $n+1$ osculating m-hedrons among $n+2$ osculating m-hedrons of an n-dimensional normal curve passes through a fixed $(n-3)$-dimensional algebraic variety of the $\{(m-n-1)^3-(m-n-2)^2\}$-th order. We may get such relations successively.

Especially when $n=2$, we have the following

COROLLARY 4. The curves of order $m-1$ determined by the vertices of any two m-latera among three complete m-latera circumscribed to a conic have $2m-5$ points in common.

This is a generalization of a theorem appearing in Piquet's book.

When $n=3$, we have the following

COROLLARY 5. The variety of order $m-2$ passing through the vertices of any three m-hedrons among four complete m-hedrons circumscribed to a twisted cubic have a curve of order $2m-7$ in common. Each of the preceding curves of order $2m-7$ determined by each set of four osculating m-hedrons among five complete m-hedrons circumscribed to the twisted cubic have $(n-4)^3-(n-5)^2$ points in common.

Special cases of the first half of this corollary are already known, i.e., the case when $m=4$ is due to Prof. T. Kubota and the case when $m=5$ is due to Prof. S. Morimoto.

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TEACHING OF MATHEMATICS

Edited by Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

STUDENT PROGRESS

Samuel S. Enser

A student can be led to water but he cannot be made to drink. He must, for some reason all his own, want to drink. A dedicated teacher may inspire his class but he cannot do the actual learning for it. For this reason the lecture method of teaching is to be frowned upon as somewhat unsound from a pedagogical viewpoint. An experienced teacher knows that preaching to a class of students is worthless folly. A good bit of energy is wasted. True, the time passes quickly for the teacher, but as the words flow faster and faster, the students sit nodding their heads with rolling eyes of sleepy approval. The student is probably deceived into thinking that this is the easiest and most economical way to knowledge, but without plucking his own fruit, he cannot taste of the tree of knowledge. It is only through exercise and much participation that intellectual progress is made. We learn by doing the philosophers say.

However, this does not mean that the teacher should throw the entire burden upon his students. He must be ready to help his students over the rough spots of the subject. He must know them individually—their strong points and their weaknesses. The needs of one are not the needs of the other. Indeed, with experience, he knows exactly where crutches will be needed to give a safe passage. He must give these boosters unstintingly.

On the other hand, the student must play the biggest part in the process. He must discover how he, himself, can study and learn most profitably and most economically with regard to time. Again the teacher can lend a helping hand, but in the end, only the student can make this discovery. Possibly he will find that the plan of attack must differ with different subjects. One thing is certain, he must not waste time in memorizing by rote. This only leads, sooner or later, to a sort of intellectual suicide. Furthermore, a course which emphasizes the remembering of a few old stale facts is certainly not worth taking. But it is amazingly possible to complete a four year course with good grades and yet be most unintellectual in everything except a few "quiz-kid" facts.

True knowledge is knowledge understood in relation to other knowledge. It must be worthy of itself. To memorize a chemical formula as one does the alphabet is certainly useless, but to find out how, or why, or when this chemical change takes place in relation to other phenomena is another matter. Thus the learning process takes on the aspects of writing a play or playing a game of chess with players which move about and containing a plot which weaves the threads of action into one continuous dramatic art. In fact, it is nothing more than the play of life itself.

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A NOTE ON COMPUTATION WITH APPROXIMATE NUMBERS

N. C. Perry and J. C. Morelock

Introduction

The study of the error in calculations with approximate numbers is a venerable one, having been of importance in engineering and physics for many years. Comparatively recently a new interest in the topic has been aroused by the advent of high-speed computing machinery in which the difficulty of estimating possible accumulation of error is one of the serious problems. [1]

At the elementary level the student of applied mathematics is taught to present his answers according to certain stereotyped rules. For example, the engineering neophyte would "round-off" the division 6.159/2.83= 2.176 ..., as 2.18. At a higher level of sophistication [2] the extremes afforded by $(6.159 \pm .0005)/(2.83 \pm .005)$ are calculated, yielding a minimum value of $6.1585/2.835 = 2.17231\ldots$, and a maximum value of $6.1595/2.825 = 2.18035\ldots$.

It is the purpose of this paper to outline a probability approach to the above type of problem. In the next section, then, we shall sketch the development of a function which associates with an error of size K (in a quotient) the probability (P) of its being exceeded.

The Probability Approach

Let us begin by assuming that E is the error in the quotient a/b caused by errors ϵ and ϵ'. Thus $E = \dfrac{a+\epsilon}{b+\epsilon'} - \dfrac{a}{b} = \dfrac{b\epsilon - a\epsilon'}{(b+\epsilon')b}$, and the probability that

E exceeds a fixed value K can be written as $P[\frac{b\epsilon - a\epsilon'}{(b+\epsilon')b} > K]$. For convenience we will think of a and b as real numbers rounded off to the nearest integer; ϵ and ϵ' will then be on the range $-\frac{1}{2}$ to $+\frac{1}{2}$. We will assume further that all values of this range have an equal likelihood of occurring, and that $a > b$.

If the inequality $E > K$ is solved for ϵ' one obtains $\epsilon' < \frac{b\epsilon - Kb^2}{Kb+a}$. Thus, with a given value of ϵ, in order for E to exceed K, ϵ' must lie on the range from $-\frac{1}{2}$ to $\frac{b\epsilon - Kb^2}{Kb+a}$. Hence the probability that simultaneously

(1) ϵ lies on a small interval $\Delta\epsilon$

and (2) ϵ' is such as to force E to exceed K

is approximately equal to the product $(\Delta\epsilon)(\frac{b\epsilon - Kb^2}{Kb+a} + \frac{1}{2})$. The essential plan of further development is to integrate this product with respect to ϵ. Thus, apparently,

$$P[E > K] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{b\epsilon - Kb^2}{Kb+a} + \frac{1}{2}\right) d\epsilon$$

Omitting lengthy manipulations (and certain modifications necessary to keep ϵ' on the range $-\frac{1}{2}$ to $+\frac{1}{2}$ for the E and K considered) we obtain the following as the function relating P and K:

$$\frac{-a-b}{2b^2+b} \leqq K \leqq \frac{b-a}{2b^2+b} \qquad P[E > K] = 1 - \frac{[(2b^2+b)K + (a+b)]^2}{8b(Kb+a)}$$

$$\frac{b-a}{2b^2+b} \leqq K \leqq \frac{a-b}{2b^2-b} \qquad P[E > K] = \frac{1}{2} - \frac{Kb^2}{Kb+a}$$

$$\frac{a-b}{2b^2-b} \leqq K \leqq \frac{a+b}{2b^2-b} \qquad P[E > K] = \frac{[(2b^2-b)K - (a+b)]^2}{8b(Kb+a)}$$

In conclusion, we illustrate the manner in which functions of this kind can be used to make probability judgments regarding error. For example, if $a/b = 8/6$ there is about one chance in twenty that the *absolute value* of the error will exceed .152, and the odds are about even as to whether it will exceed .06.

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[1] Householder, A. S. Principles of Numerical Analysis, New York, 1953.
[2] Scarborough, J. B. Numerical Mathematical Analysis, Baltimore, 1950.

Alabama Polytechnic Institute

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

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I note with interest on page 158 of the Jan-.Feb. issue of *Mathematics Magazine,* the derivation of the director circle of an ellipse.

If the method of C. Smith (*Conic Sections,* p. 255) is applied to the central conic

$$ax^2 + 2hxy + by^2 + c = 0,$$

it is seen that a pair of tangents from (x', y') is

$$(ax^2+2hxy+by^2+c)(ax'^2+2hx'y'+by'^2+c) = \{axx'+h(xy'+yx')+byy'+c\}^2 .$$

These lines are perpendicular if the sum of the coefficients of x^2 and y^2 is zero, that is, if

$$(a+b)(ax'^2+2hx'y'+by'^2+c) = (ax'+hy')^2+(hx'+by')^2 .$$

Hence the director circle being the locus of (x', y') is

$$(ab-h^2)(x^2+y^2) + (a+b)c = 0; \qquad (1)$$

of the central conic

$$ax^2 + 2hxy + by^2 + c = 0. \qquad (2)$$

for the ellipse

$$B^2x^2 + A^2y^2 - A^2B^2 = 0,$$

clearly $a = B^2$, $h = 0$, $b = A^2$, $c = -A^2B^2$, and these values placed in (1) give

$$x^2 + y^2 - (A^2 + B^2) = 0$$

as found before.

Paul D. Thomas
Eglin Air Force Base, Florida

Concerning the step problem, *Long Short Lines*, Vol. 30, page 294, I have noted:

Given a step function

$S \equiv$ 

the step function $\to$ a straight line as a limit.

If a function is bounded and uniformly continuous we can write

$$f(x) = P_0 + (P_1 - P_0) + (P_2 - P_1) + \dots .$$

The series converges uniformly and absolutely.

Therefore the derivative

$$P_0' + (P_1' - P_0') + (P_2' - P_1') + \dots$$

converges [Borel. *Leçons sur les Fonctions de Variables Rêeles et les Dêveloppements en Séries de Polynomes*].

What about the derivative of the step function? Borel's theorem seems to indicate that a limit exists but how can a line with an infinite number of sharp points have a derivative?

I leave the paradox to you.

L. B. Robbins

* * * * * * * * * *

This comment concerns the apparent paradox in the paper "*On a Certain Problem in Mechanics*", November-December 1956, and the comment on this paper, May-June 1957.

My comment is: is there a paradox? The problem asks: *how far* from the starting place does the object come to rest. The problem does not ask *when* this event will take place.

If I consider the series $\sum_0^\infty 1/2^n$, the fact that I may spend one second writing each term does not keep $\sum_0^\infty 1/2^n = 2$ from being true. That the "physical" obtention of a result is impossible does not invalidate the correctness of a result. The mere fact of a frictionless plane precludes a physical obtention of the result in the original problem.

I remember meeting a similar type of problem in a course on non-linear vibrations. What force is needed to move a mass m at the end of a massless rod of length l pivoting at the other end on a frictionless shaft for the mass m (originally hanging at rest) to complete a half revolution so that it remains in equilibrium (unstable) in its new position. In this case equally, the time needed is ∞.

In other words, the problem, mathematically speaking, is straightforward; and has a straightforward answer. Observing an ideal condition (frictionless plane) and wanting a practical result is the only thing that gives birth to a paradox.

Philip B. Jordain

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A CORRECTION AND GENERALIZATION OF NEUSTADT'S LAW

In MATHEMATICS MAGAZINE (November-December 1956), Paul Schillo presented as a jest, a law credited to an I. Neustadt.

The series $.2-1+2-2+1-.2$ is called "Neustadt's basic series," or simply "Neustadt's base." According to Mr. Schillo, "The first arithmetic progression which Neustadt used was $1, 2, 3, 4, 5, 6$. He produced the derived series (or derivative) $.2-2+6-8+5-1.2$ by multiplying the successive terms in the base by the respective terms in the progression and observed that it too had a zero sum. He used the arithmetic series as an operator on his first derivative to obtain a second derivative which was zero sum also. The arithmetic series operating on his second derivative produced a third derivative which was also zero sum."

According to Mr. Schillo's account, "He (Neustadt) then copyrighted his famous law on Flag Day 1932. Namely: For any arithmetic progression all derivatives of any order of the base $.2-1+2-2+1-.2$ are zero sum."

The author now offers the following results concerning so called Neustadt derivatives.

I. Arithmetic Progression $\sum_{p=0}^{5} (1+pd)$

(a) coefficent of d^0 in n'th derivative

$= \text{Neustadt's Base} = 0$

(b) coefficient of d^x $(x \neq 0)$ in n'th derivative

$= C(n,x)[-1+2(2)^x-2(3)^x+4^x-5^{x-1}]$

which vanishes if $x = 1, 2, 3, 4$

but not if $x = 5$

II. Plane Figurate Progression $\sum_{p=0}^{5} [(1+p) + \frac{p(p+1)}{2}d]$

(a) coefficient of d^0 in n'th derivative

$= .2-2^n+2(3)^n-2(4)^n+5^n-.2(6)^n$

which has zeros of $1, 2, 3, 4$, but not 5.

(b) coefficient of d^x $(x \neq 0)$ in the n'th derivative

$= C(n, x)\ [-2^{n-x} + 2(3)^n - 2(4)^{n-x}(6)^x + 5^{n-x}10^x - .2(6)^{n-x}(15)^x]$

which has zeros of (1, 1), (2, 1), (2, 2), (3, 1) but not (3, 2).

III. Solid (three dimensional) Figurate Progression

$$\sum_{p=0}^{5} \frac{(p+1)(p+2)}{6}(3+pd)$$

(a) coefficient of d^0 in n'th derivative

$= .2 - 3^n + 2(6)^n - 2(10)^n + 15^n - .2(21)^n$

which has zeros of 1 and 2 but not 3

(b) coefficient of d^x $(x \neq 0)$ in the n'th derivative

$= C(n, x)\ [-3^{n-x} + 2(6)^{n-x}(4)^x - 2(10)^n + 15^{n-x}20^x - .2(21)^{n-x}(35)^x]$

which has zero of (1, 1) but not (2, 1)

IV. Four Dimensional Figurate Progression

$$\sum_{p=0}^{5} \frac{(p+1)(p+2)(p+3)}{24}(4+pd)$$

(a) coefficient of d^0 in n'th derivative

$= .2 - 4^n + 2(10)^n - 2(20)^n + 35^n - .2(56)^n$

which has a zero of 1 but not 2.

(b) coefficient of d^x $(x \neq 0)$ in n'th derivative

$= C(n,x)[-4^{n-x} + 2(10)^{n-x}(5)^x - 2(20)^{n-x}(15)^x + 35^n - .2(56)^{n-x}(70)^x]$

which has a zero of (1,1).

V. Q Dimensional Figurate Progression

$$\sum_{p=0}^{5} \frac{1}{q!}\frac{(q+p-1)!}{p!}(q+pd)$$

as being concerned only with the first derivative.

(a) coefficient of d^0

$$= .2 + q^2 - \frac{q(q+1)(q+2)}{3} + \frac{q(q+1)(q+2)(q+3)}{4!} - .2\frac{q(q+1)(q+2)(q+3)(q+4)}{5!}\cdot$$

which has zeros (as we of course already know) of 1, 2, 3, 4 but (as we now find out) does not have a zero of 5.

(b) coefficient of d

$$= -1 + 2(q+1) - (q+1)(q+2) + \frac{(q+1)(q+2)(q+3)}{3!} - .2\frac{(q+1)(q+2)(q+3)(q+4)}{4!}$$

which has (as we already know) zeros of 1, 2, 3, and 4, but not of 5.

We now give a revision of "Neustadt's Law."

Using a figurate progression in which the first term is unity, we can be certain of the following number of vanishing Neustadt derivatives: Four if operator is arithmetic; two if operator progression is plane; one if operator progression is of dimension three or four; and none if the operator progression is of dimension greater than four.

W. W. Funkenbusch
Michigan College of Mining and Technology

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Linear Algebra for Undergraduates. By D. C. Murdoch, John Wiley & Sons, Inc., New York, 1957. 239 pages. $5.50.

Elementary in treatment, the new book keeps abstract ideas to a minimum, while geometric motivations and applications for the abstract algebraic theorems are stressed. Chapters are devoted to: vectors and vector spaces; matrices, rank, and systems of linear equations; further algebra of matrices; further geometry of real vector spaces; transformations of coordinates and linear transformation in a vector space; similar matrices and diagonalization theorems; reduction of quadratic forms; and vector spaces over the complex field.

Largely written for the student, *Linear Algebra for Undergraduates* contains material of use to physicists, engineers, mathematicians, and statisticians. Dr. Murdoch avoids abstract algebra in his treatment of the properties of matrices and quadratic forms of importance to physicists, and includes some applications to differential equations and physical problems. The author also recognizes that matrix methods are being used increasingly in engineering, especially in the theory of vibrations, elasticity, and electrical networks. The relevant algebra is treated here with simplicity.

Richard Cook

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Elements of Algebra. Levi, Second Edition, Chelsea Publishing Company, New York.

Elements of Algebra is a text for the required one-semester mathematics course given in the School of General Studies of Columbia University. It undertakes to provide a coherent basis for the further study of mathematics and in addition to give the student whose chief interests lie elsewhere an acquaintance with general mathematical problems and procedures. No previous training in algebra is presupposed.

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Games and Decisions: An Introduction and Critical Survey. By R. Duncan Luce and Howard Raiffa, John Wiley & Sons, Inc., New York, 1957. 509 pages. $8.75.

The authors communicate the central ideas and results of game theory and related decision-making models with a minimum of technical mathematics. Directing their work to the attention of behavioral scientists, the two mathematician-authors have colored their critical discussion and examples by a social science point of view.

"Our primary topic," Luce and Raiffa state in their preface, "can be viewed as the problem of individuals reaching decisions when they are in conflict with other individuals and when there is risk involved in the outcomes of their choices." In clarifying this problem, the authors first supply a general and intuitive description in the first chapter. This is followed by discussions of utility theory, extensive and normal forms, two-person zero-sum games, two-person non-zero-sum non-cooperative games, and two-person cooperative games. Chapters 7 through 12 cover the theories of games with more than two players: theories of n-person games in normal form, characteristic functions, solutions, psi-stability, reasonable outcomes and value, and applications of n-person theory.

Richard Cook

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

327. *Proposed by Chih-yi Wang, University of Minnesota.*

Show that the curve $x^6 + y^6 - 18(x^4 + y^4) + 81(x^2 + y^2) - 108 = 0$ consists of two ellipses and a circle.

328. *Proposed by Walter B. Carver, Cornell University.*

Let AB and CD be parallel lines cut by a transversal at the points E and F respectively. Lines EM and EN trisect angle FEB and FL and FO trisect angle EFD where angle $FEM <$ angle FEN and angle $EFL <$ angle EFO. Let P be the intersection of EM and FL while Q is the intersection of EN and FO. Through P draw a line parallel to FQ cutting EQ at G and a line parallel to EQ cutting FQ at H. The line GH cuts AB at J and CD at K. Show that $JG = GH = HK$.

329. *Proposed by D. A. Breault, Waltham, Massachusetts.*

Given X_i, normally distributed with fixed mean μ, and some $\epsilon > 0$, determine $\sigma = \sigma(N, \epsilon)$ such that the following holds:

$$P\left[N\left\{\sum_{i=1}^{N} X_i^2\right\} - \left\{\sum_{i=1}^{N} X_i\right\}^2 < \epsilon\right] \geqq .95$$

330. *Proposed by M. N. Gopalan, Mysore, India.*

Solve

$$x + y + z = a$$
$$x^3 + y^3 + z^3 = b$$
$$x^5 + y^5 + z^5 = c$$

331. *Proposed by Paul M. Pepper, Ohio State University.*

Show that if n is an integer, then $\left[2^n + (-1)^{n+1}\right]/3$ is zero, an odd integer or an odd integer divided by a positive integral power of 2 according as n is zero, positive or negative.

332. *Proposed by Norman Anning, Alhambra, California.*

Prove that there is no polynomial of degree 22 which is an exact divisor of $x^{45} + 1$.

333. *Proposed by Barney Bissinger, Lebanon Valley College.*

Find a closed expression for

$$F(x,y,z) = \frac{x^2}{y}\left\{(z-1) + \left[2z - (1+2)\right]\frac{y-x}{y} + \left[3z-(1+2+3)\right]\left(\frac{y-x}{y}\right)^2 + \ldots + \left[nz-(1+2+\ldots+n)\right]\left(\frac{y-x}{y}\right)^{n-1}\right\}$$

SOLUTIONS

Late Solutions

301. *M. N. Gopalan, Mysore, India; J. M. Gandhi, Jain Engineering College, Panchkoola, India; K. L. Duggal, Jain Engineering College, Panchkoola, India.*

Congruent Figures

306 [May 1957] *Proposed by V. F. Ivanoff, San Francisco, California.*

A student drew a line segment, AC, four inches long. He constructed the mid point B so that $AB = BC = 2$ inches. He reasoned that if he removed the segment BC he would have left a segment which has no end point and is clearly not congruent with BC. This seemed to contradict his hypotheses that B was the mid point of AC.

Wherein lies the student's dilemma? What would be the case if AC were replaced by a circle?

Comments by Lawrence A. Ringenberg, Eastern Illinois State College.

Solution **I.** The mid point belongs to AB as much as to BC. The segment BC, C included and B excluded, is congruent in an intuitive sense to the segment AB, A included and B excluded. And, of course, the closed segments AB and BC, as well as the open segments AB and BC, are congruent in the same intuitive sense.

Solution **II.** B is the mid point of AC if the lengths of AB and BC are equal. Whether the end points of segments AB and BC are included or not has no effect on their lengths.

Solution **III.** According to the Hilbert axiom on congruence, if AB is a segment, there is a unique point C on line AB such that B lies between A and C and such that segment AB is congruent to segment BC. Whether or not the end points are included with the segments is no part of the congruence concept. Hilbert's axiom provides a mathematical basis for the notion of congruent segments. Removing segments (moving geometric figures about without changing their size or shape) is no part of this theory.

Appendix I. If AC is replaced by a circle, then it takes two points, B and B', of the circle to separate the circle into congruent arcs BAB' and BCB'. As in Solution I, these arcs are congruent in an intuitive sense if both are closed, if both are open, or if both are semi-open.

Appendix II. The arcs BAB' and BCB' are congruent if they have the same length. Whether the end points are included with the arc has no effect on their lengths.

Appendix III. We could invent a Hilbert type axiom for congruent arcs on a circle in order to provide an analogue for Solution III.

Comment I. If AC is replaced by a circular disc, then a "mid point" which separates it into two congruent semicircular regions is not one point or two points but all the points on any diameter of the circle (many other continuous arcs might be admitted here). This "mid point" does not have a length (one-dimensional measure) of 0, but it does have an area (two-dimensional measure) of 0.

Comment II. If AC is replaced by a sphere and its interior, then a "mid point" which separates it into two congruent regions is the set of points common to any plane through the center of the sphere and the sphere itself (other continuous surfaces might be admitted). This "mid point" has three-dimensional measure 0 and hence, if it is included with one of the regions into which it separates the sphere plus its interior, it does not increase the measure of that region.

Comment III. Any student who reasons himself into this dilemma should be encouraged to study mathematics.

Also solved by D. A. Breault, Waltham, Massachusetts.

The Sum of Two Radicals

307 [May 1957] *Proposed by E. P. Starke, Rutgers University.*

If a number n is of the form $n = \sqrt{a+1} + \sqrt{a}$ with a rational, prove that every integral power of n is of the same form.

I. *Solution by W. A. Al-Salam and L. Carlitz, Duke University.*
Put

$$(1) \qquad (\sqrt{a+1} + \sqrt{a})^{2n} + (\sqrt{a+1} - \sqrt{a})^{2n} = 4a_n + 2$$

If a is rational it is clear that a_n is also rational for all positive integral n. Also it follows from (1) that

$$(\sqrt{a+1}+\sqrt{a})^{2n}+2+(\sqrt{a+1}-\sqrt{a})^{2n}=4a_n+4\ ,$$

$$(\sqrt{a+1}+\sqrt{a})^{2n}-2+(\sqrt{a+1}-\sqrt{a})^{2n}=4a_n\ ,$$

which imply

$$(\sqrt{a+1}+\sqrt{a})^{n}+(\sqrt{a+1}-\sqrt{a})^{n}=2\sqrt{a_n+1}\ ,$$

$$(\sqrt{a+1}+\sqrt{a})^{n}-(\sqrt{a+1}-\sqrt{a})^{n}=2\sqrt{a_n}\ ,$$

and therefore

$$(\sqrt{a+1}+\sqrt{a})^{n}=\sqrt{a_n+1}+\sqrt{a_n}.$$

This evidently proves the assertion.

Note that if a is integral then a_n is also integral. Indeed (1) implies

$$2a_n+1=\sum_{r=0}^{n}\binom{2n}{2r}(a+1)^{n-r}a^r\ . \tag{2}$$

But

$$\binom{2n}{2r}\equiv\binom{n}{r}\qquad(\text{mod } 2),$$

so that

$$2a_n+1\equiv\sum_{r=0}^{n}\binom{r}{n}(a+1)^{n-r}a^r\equiv(2a+1)^n\equiv 1\qquad(\text{mod } 2).$$

Thus the right member of (2) is odd and a_n is therefore integral.

As an extension of the original problem we now prove the following Theorem. The equation

$$(\sqrt{a+1}+\sqrt{a})(\sqrt{b+1}+\sqrt{b})=\sqrt{c+1}+\sqrt{c} \tag{3}$$

is solvable in positive integers a, b, c if and only if there exist integers k, m, n such that

$$\sqrt{a+1}+\sqrt{a}=(\sqrt{k+1}+\sqrt{k})^m\ ,$$

$$\sqrt{b+1}+\sqrt{b}=(\sqrt{k+1}+\sqrt{k})^n\ .$$

The sufficiency is already proved. To prove the necessity, we have first from (3) by taking reciprocals

$$(\sqrt{a+1} - \sqrt{a})(\sqrt{b+1} - \sqrt{b}) = \sqrt{c+1} - \sqrt{c}\ .$$

Taken with (3) this yields

$$a(b+1) + (a+1)b + 2\sqrt{a(a+1)\,b(b+1)} = c\ .$$

Thus it is necessary that the number

$$(4) \qquad a(a+1)\,b(b+1)$$

be a square.

Now let a be fixed and let f^2 denote the greatest square dividing $a(a+1)$, so that

$$a(a+1) = f^2 D\ ,$$

where D is square-free and > 1. Thus the necessary condition is

$$b(b+1)\,D = x^2\ ,$$

which implies $(x = Dy)$

$$b(b+1) = Dy^2\ ,$$

$$(5) \qquad (2b+1)^2 - 4Dy^2 = 1\ .$$

By the theory of the Pellian equation (see for example Chystral's *Algebra,* Vol. 2) the equation (4) has a fundamental solution u, v such that all positive solutions are given by

$$(6) \qquad (u + 2v\sqrt{D})^n = u_n + 2v_n\sqrt{D}\ .$$

Thus there is an integer n such that

$$2b + 1 = u_n;$$

since u_n is necessarily odd, this determines b as an integer.

In the next place it is easily verified that

$$u_n + 2v_n\sqrt{D} = (\sqrt{b+1} + \sqrt{b})^2\ ,$$

$$u + 2v\sqrt{D} = (\sqrt{k+1} + \sqrt{k})^2,$$

where $2k+1 = u$. Thus (6) becomes

$$(\sqrt{k+1} + \sqrt{k})^n = \sqrt{b+1} + \sqrt{b}.$$

If we now fix b, we find in exactly the same way that there is an integer m such that

$$(\sqrt{k+1} + \sqrt{k})^m = \sqrt{a+1} + \sqrt{a};$$

since (4) is a square, the number D is the same in both cases. This completes the proof of the theorem.

We remark that the theorem is not true when a, b, c in (3) are allowed to assume fractional values. A counter example is furnished by

$$(1+\sqrt{2})\left(\sqrt{\frac{1}{7}}+\sqrt{\frac{8}{7}}\right)= \sqrt{\frac{18}{7}} +\sqrt{\frac{25}{7}}$$

Assume that each factor is a power of same

$$\sqrt{k} + \sqrt{k+1}. \tag{7}$$

Then at least one of the exponents is even, and the corresponding factor is the square of a number of the form (7). This is however not the case, as is easily verified.

II. *Solution by Calvin A. Rogers, Colorado State University.*

Define a number x by the equation

$$x = \log_e \sqrt{a+1} + \sqrt{a}).$$

Note that x is > 0, that $x = \log_e n$, and that accordingly $n = e^x$. Also note that $a = \sinh^2 x$.

Let p be any positive integer. Then $n^p = e^{px} = \cosh(px) = \sinh(px)$. As $\cosh(px)$ is always positive for real x, $\cosh(px) = |\cosh(px)|$; since x and p are positive, $\sinh(px)$ is positive, and accordingly $\sinh(px) = |\sinh(px)|$.

So $n^p = |\cosh(px)| + |\sinh(px)| = \sqrt{\sinh^2(px)+1} = \sqrt{\sinh^2(px)}$ has the same form as n itself.

It remains to show that if $\sinh^2 x$ (namely, a) is rational, so is $\sinh^2(px)$. But $\sinh^2(px) = \frac{1}{2}(\cosh(2px) - 1)$. But $\cosh(2px) = T_{2p}(\cosh x)$, where (following the notation of Lanczos', "Applied Analysis," pp. 178-179)

$T_{2p}(\cosh x)$ denotes the Chebyshev polynomial of the first kind and degree $2p$, with its argument replaced by $\cosh x$. Since $2p$ is an even number, this polynomial in $\cosh x$ contains only *even* powers, i.e., it contains only powers of $\cosh^2 x$. But $\sinh^2 x$ being rational, so is $\cosh^2 x = 1 + \sinh^2 x$, and so, therefore, is $T_{2p}(\cosh x)$. Hence, finally, so is $\sinh^2(px)$.

Also solved by George Bergman, Junior High School 246, Brooklyn, New York; W. B. Carver, Cornell University; Irving A. Dodes, Bronx High School of Science, New York; Harry M. Gehman, University of Buffalo; Herbert R. Leifer, Pittsburgh, Pennsylvania; F. D. Parker, University of Alaska; Arne Pleyet, Trollhattan, Sweden; Harry D. Ruderman, Polytechnic Institute of Brooklyn; Harry Schor, Far Rockaway High School, New York; Chih-yi Wang, University of Minnesota and the proposer.

Wang pointed out an analagous problem in E 950, *American Mathematical Monthly,* Vol. 58 (1951), p. 566.

The proposer mentioned that the problem was an outgrowth of a remark by Norman Anning.

Perfect Cubes

309 [May 1957] *Proposed by Victor Thebault, Tennie, Sarthe, France.*

Determine the perfect cubes terminated on the right by nine digits 1 (or by nine digits 3).

Solution by C. W. Trigg, Los Angeles City College. From *Barlow's Tables* we quickly find the cube of the only four-digit number whose cube terminates in four ones is $(8471)^3 = 607\ 860\ 671\ 111$. Now $(a\ 8471)^3 = (a\ 10^4)^3 + 3(a\ 10^4)^2(8471) + 3(a\ 10^4)(8471)^2 + (8471)^3$. Since only the last two terms of the expansion can affect the fifth place from the right, $3(a)(1)^2 + 7 = b \cdot 1$, so $a = 8$. Proceeding in like manner without complete expansion we find that a 2 may be prefixed to the 8 to obtain $(288471)^3 = 24\ 005\ 263\ 647\ 111\ 111$. Again, working with the last two terms of the expansion of the cube of a binomial we immediately find 8, 6, and 3 to be the successive prefixes necessary to obtain the unique nine-digit solution: $(368288471)^3 = 49\ 953\ 321\ 584\ 048\ 960\ 111\ 111\ 111$. It follows that the cubes of all numbers terminating in 368 288 471 will terminate in nine 1's.

In like manner it may be shown that all other cubes terminating in nine like digits derive from

$(385\ 446\ 477)^3 = 57\ 265\ 392\ 488\ 653\ 305\ 333\ 333\ 333,$
$(899\ 660\ 753)^3 = 728\ 175\ 940\ 489\ 979\ 486\ 777\ 777\ 777,$
$(236\ 576\ 942)^3 = 13\ 240\ 891\ 943\ 284\ 634\ 888\ 888\ 888,$
$(486\ 576\ 942)^3 = 115\ 200\ 555\ 682\ 838\ 157\ 888\ 888\ 888,$
$(736\ 576\ 942)^3 = 399\ 626\ 572\ 672\ 391\ 680\ 888\ 888\ 888,$
$(986\ 576\ 942)^3 = 960\ 268\ 942\ 911\ 945\ 203\ 888\ 888\ 888,$
$(999\ 999\ 999)^3 = 999\ 999\ 997\ 000\ 000\ 002\ 999\ 999\ 999,$

and the cube of any number terminating in three zeros.

Also solved by Leon Bankoff, Los Angeles, California; M. Pembroke and D. A. Breault (Jointly), Waltham, Massachusetts; E. P. Starke, Rutgers University and the proposer.

Elliptic Integrals

310. [May 1957] *Proposed by Chih-yi Wang, University of Minnesota.*

Show that

$$\int_0^{\pi/2} \sqrt{\cos x}\, dx = \int_0^{\pi/2} \sqrt{\sin x}\, dx = 2\sqrt{2}\,E(1/\sqrt{2}) - \sqrt{2}\,F(1/\sqrt{2})$$

where $E(k)$ and $F(k)$ are elliptic integrals defined respectively by,

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 t}\, dt, \quad F(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

I. *Solution by Karl O. Malmquist Jr., University of California Radiation Laboratory.*

Let $\phi = \pi/2 - \beta$

then

$$\int_0^{\pi/2} \sqrt{\sin\phi}\, d\phi = \int_0^{\pi/2} \sqrt{\cos\beta}\, d\beta$$

let $\cos\beta = \cos^2\alpha$ $\qquad d\beta = \dfrac{-2\cos\alpha\, d\alpha}{\sqrt{1+\cos^2\alpha}}$. Then

$$\int_0^{\pi/2} \sqrt{\cos\beta}\, d\beta = \int_0^{\pi/2} \frac{2\cos^2\alpha}{\sqrt{1+\cos^2\alpha}}\, d\alpha =$$

$$2\left\{\int_0^{\pi/2} \frac{1+\cos^2\alpha}{\sqrt{1+\cos^2\alpha}}\, d\alpha - \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1+\cos^2\alpha}}\right\}$$

$$2\left\{\int_0^{\pi/2} \sqrt{1+\cos^2\alpha}\, d\alpha - \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1+\cos^2\alpha}}\right\}$$

$$= 2\left\{\sqrt{2}\int_0^{\pi/2}\sqrt{1-1/2\sin^2\alpha}\,d\alpha - \frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-1/2\sin^2\alpha}}\right\}$$

$$= \sqrt{2}\left\{2\int_0^{\pi/2}\sqrt{1-1/2\sin^2\alpha}\,d\alpha - \int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-1/2\sin^2\alpha}}\right\}$$

$$= \sqrt{2}\left\{2E(1/\sqrt{2},\ \pi/2) - F(1/\sqrt{2},\ \pi/2)\right\}$$

where $E(1/\sqrt{2},\ \pi/2)$ = elliptic integral of the second kind; $F(1/\sqrt{2},\ \pi/2)$ = elliptic integral of the first kind.

II. *Comment by John L. Brown Jr., Pennsylvania State University.* This integral also has a known representation in terms of gamma functions [e.g. Grobner–Hofreiter–"Integraltafel," Volume II, page 97]

$$\int_0^{\pi/2} \sin x\,dx = \frac{\sqrt{\pi}\,\Gamma(3/4)}{2\,\Gamma(5/4)}$$

The transformation used above can also be employed to prove

$$\int_0^{\pi/2}\frac{dx}{\sqrt{\cos x}} = \int_0^{\pi/2}\frac{dx}{\sqrt{\sin x}} = \sqrt{2}F(1/\sqrt{2}) = \frac{\sqrt{2}}{4\sqrt{\pi}}\left[\Gamma(1/4)\right]^2$$

From these results, several interesting identities can be derived relating the complete elliptic integrals having argument $\frac{1}{\sqrt{2}}$ to gamma functions.

III. *Comment by D. A. Breault, Waltham, Massachusetts.* The problem can be generalized to arrive at the integral

$$\int_{u_1}^{u_2}\sqrt{A+B\cos(u-b)}\,du = \left[\frac{2(A-B)}{\sqrt{2B}}F(C,w) + 2\sqrt{2B}\,E(C,w)\right]_{w_1}^{w_2}$$

where $C = \sqrt{\frac{A+B}{2B}} < 1$ and $\sin w_2 = 1/C\sin\frac{u_2-b}{2}$; $\sin w_1 = 1/C\sin\frac{u_1-b}{2}$.

Also solved by F. D. Parker, University of Alaska; A. K. Rajagopal, Indian Institute of Science, Bangalore, India; Calvin A. Rogers, Colorado State University; Dale Woods and Harold Johnson (Jointly) Memphis State

University and the proposer.

The Coefficients of $\frac{\sinh x}{\sin x}$

311. [May 1957] *Proposed by J. M. Gandhi, Lingraj College, Belgaum, India.*

Prove the following relation.

$$(2n+1)!!\ B_{2n} = (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -\binom{3}{3} & \binom{3}{1} & \cdots & 0 \\ 1 & \binom{5}{5} & -\binom{5}{3} & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & (-1)^n\binom{2n+1}{2n+1} & (-1)\binom{2n+1}{2n-3} & \cdots & -\binom{2n+1}{3} \end{vmatrix}$$

where $(2n+1)!! = 1\cdot 3\cdot 5 \ldots (2n+1)$ and B_{2n} are the coefficients in the expansion of $\frac{\sinh n}{\sin n}$

Solution by Chih-yi Wang, University of Minnesota. It is known that

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

define (not in the usual sense) their quotient by $\sum_{n=0}^{\infty} \frac{B_{2n}x^{2n}}{(2n)!}$. Then we have by equating the corresponding coefficients,

$$\begin{aligned} B_0 &= 1 \\ -B_0 + \binom{3}{1}B_2 &= 1 \\ B_0 - \binom{5}{3}B_2 + \binom{5}{1}B_4 &= 1 \\ -B_0 + \binom{7}{5}B_2 - \binom{7}{3}B_4 + \binom{7}{1}B_6 &= 1 \\ \cdots & \quad \cdot \\ \cdots & \quad \cdot \\ \cdots & \quad \cdot \end{aligned}$$

Solving for B_{2n} by means of Cramer's rule, we get the required result by interchanging the last column of the determinant in the numerator to the first column and noting the value of the denominator is precisely $(2n+1)!!$.

If B_{2n} are defined in the ordinary sense, we need a factor $(2n)!$ to the left of the given expression.

Also solved by the proposer who pointed out the relation of the problem to the article by L.Carlitz on The Coefficients of $\sinh x/\sin x$ appearing in this magazine Vol. 29 No. 4, March-1956, Page 193.

Fibonacci Numbers

312. [May 1957] *Proposed by A. S. Gregory, Chicago, Illinois.*

Evaluate.

$$\sum_{m=4}^{n}\left\{\sum_{i=0}^{[\frac{a-1}{2}]}\binom{a-1-i}{i}\sum_{j=0}^{[\frac{m-a}{2}]}\binom{m-a-j}{j}+\sum_{k=0}^{[\frac{a-2}{2}]}\binom{a-2-k}{k}\sum_{h=0}^{[\frac{m-a-1}{2}]}\binom{m-a-1-h}{h}\right\}$$

for $2 < a < m < n = 4, 5, 6, \ldots$.

Solution by the proposer. The nth Fibonacci number is given by the expression

$$F_n = \sum_{m=0}^{(\frac{n-1}{2})}\binom{n-1-m}{m}.$$

Futhermore we know that

$$F_n = F_a F_{n-a+1} + F_{a-1}F_{n-a},\ 1 < a < (n-1)$$

Therefore the proposed summation equals

$$\sum_{m=4}^{n} F_m = F_{n+2} - 5 .$$

Generalization of Problem 283

283. [September 1956] Proposed by Jack Winter and Richard C. Kao, The Rand Corporation, Santa Monica, California.

Editor's note: This department encourages heuristic treatments of problems and generalizations.

Generalization by Paul M. Pepper, Ohio State University. Problem No. 283 proposed in Vol. 30, No. 1, September-October 1956 requests the proof of a formula which is a special case of the relation

$$\sum_{a_n=0}^{N} \sum_{a_{n-1}=0}^{a_n} \cdots \sum_{a_1=0}^{a_2} 1 = \binom{N+n}{n} \tag{1}$$

In (1) no restriction other than $N \geqq 0$, $n \geqq 1$ need be placed onthe integers N and n. The proof of this more general formula is quite simple.

First, because of the nature of the summation we note that each set of values of the indices appearing in the summation must satisfy

$$N \geqq a_n \geqq a_{n-1} \geqq a_{n-2} \cdots \geqq a_2 \geqq a_1 \geqq 0 \ .$$

Remark A. Thus when $N = 0$ and n is arbitrary, the summation reduces to the single term 1. Since $\binom{0+n}{n} = 1$, the relation is valid when $N = 0$, for all n.

Remark B. For $n = 1$, the summation reduces to

$$\sum_{a_1=0}^{N} 1 = N + 1 \ .$$

Since $\binom{N+1}{1} = N + 1$, the relation is valid for all N when $n = 1$.

The Remarks A and B serve as a starting point for a proof usinga double induction on n and N. Since Remark A shows that the relation is valid for $N = 0$ regardless of the value of n, the double induction reduces to a very simple case in which the transition from N to $N + 1$ is effected for a given n on the basis of the relation being valid for $n - 1$ and all values of N.

Let fixed values n and N be given and let it be assumed that the relation holds for $n - 1$ coupled with all values of the last upper limit of summation and for the given n coupled with the given value N. To show that the relation is valid for n coupled with $N + 1$, we write

$$\sum_{a_n=0}^{N+1} \sum_{a_{n-1}=0}^{a_n} \cdots \sum_{a_1=0}^{a_2} 1 = \sum_{a_n=0}^{N} \sum_{a_{n-1}=0}^{a_n} \cdots \sum_{a_1=0}^{a_2} 1 + \sum_{a_{n-1}=0}^{N+1} \cdots \sum_{a_1=0}^{a_2} 1 \ .$$

By inductive hypothesis the first term on the right equals $\binom{N+n}{n}$ and the second term on the right equals $\binom{N+n}{n-1}$. But, as is readily shown, $\binom{N+n}{n}+\binom{N+n}{n-1} = \binom{(N+1)+n}{n}$, thus the left side has the desired value if the inductive hypothesis holds. This completes the induction for the given n assuming there is a suitable initial value of N for which the theorem is valid. Remark A furnishes such an initial value, $N = 0$, for all values of n. Thus, if the theorem has been proved for a given $n - 1$ and all values of N, then it holds for the next value n and all values of N. This completes the transitional theorem to pass from $n-1$ to n. It remains only to show that the theorem holds for an initial value of n with all values of N. This is the role of Remark B, for it says the theorem is valid when $n=1$ irrespective of the value of N. We may then conclude that the theorem is valid for all integers $n \geqq 1$ and $N \geqq 0$.

A corollary of the first general result is the relation

$$\sum_{a_n=0}^{N} \sum_{a_{n-1}=0}^{a_n} \cdots \sum_{a_1=0}^{a_2} \binom{a_1+r}{r} = \binom{N+n+r}{n+r} \tag{2}$$

for each integer $r \geqq 0$, assuming that $\binom{0}{0} = 1$. This follows since the summand can be rewritten in the form

$$\binom{a_1+r}{r} = \sum_{b_r=0}^{a_1} \sum_{b_{r-1}=0}^{b_r} \cdots \sum_{b_1=0}^{b_2} 1,$$

with the result that the left member of (2) becomes a sum of the type of the left member of (1) involving $r+n$ summations instead of n summations.

A relation similar to (2) can be derived in which the summation index appears in both the upper and lower rows of the left member. This relation is

$$\sum_{a_m=0}^{M} \sum_{a_{m-1}=0}^{a_m} \cdots \sum_{a_1=0}^{a_2} \binom{s+a_1}{a_1} = \binom{s+m+M}{M} \tag{3}$$

The proof of this relation is considerably simpler than that of (1). Note that the left member equals

$$\sum_{a_m=0}^{M} \sum_{a_{m-1}=0}^{a_m} \cdots \sum_{a_2=0}^{a_3} \left[\sum_{a_1=0}^{a_2} \binom{s+a_1}{a_1} \right]$$

The term in brackets can, by a simple induction be shown to equal

$\binom{s+1+a_2}{a_2}$. Since this has the same form as the summand except that $s+1$ replaces s and a_2 replaces a_1, it follows by an induction that the two sides of (3) are equal.

But by (1) the right member of (3) equals the left member of (1) with n replaced by M, and N replaced by $s+m$, whereas the summand on the left side of (3), is, by (1), equal to the left side of (1) with N replaced by s and n replaced by a_1, whence (dropping the subscript on the running index a_1) we obtain the relation

$$\sum_{a_M=0}^{s+m}\sum_{a_{M-1}=0}^{a_M}\cdots\sum_{a_1=0}^{a_2}1=\sum_{a_m=0}^{M}\sum_{a_{m-1}=0}^{a_m}\cdots\sum_{a=0}^{a_2}\left(\sum_{c_a=0}^{s}\sum_{c_{a-1}=0}^{c_a}\cdots\sum_{c_1=0}^{c_2}1\right), \tag{4}$$

which is effectively a method of reversing the order of summation; that is, the number M of sums on the left side becomes the number of terms less one in the last or outer sum on the right side, and m which represents a partial measure of the number of terms in the last or outer sum on the left side becomes the number of sums in the outer portion of the right side. Moreover, a_1 (or a) which represents indices of the various terms in the innermost sum of the left side becomes, for each of its values, the number of sums inside the parentheses on the right side.

Since the right side of (3) is symmetric in m and s we may interchange them in the left side of (3) to obtain

$$\sum_{a_s=0}^{M}\sum_{a_{s-1}=0}^{a_s}\cdots\sum_{a_1=0}^{a_2}\binom{m+a_1}{a_1}=\binom{s+m+M}{M}, \tag{3'}$$

and from (3) and (3′) we can obtain

$$\sum_{a_m=0}^{M}\sum_{a_{m-1}=0}^{a_m}\cdots\sum_{a_1=0}^{a_2}\binom{s+a_1}{a_1}=\sum_{a_s=0}^{M}\sum_{a_{s-1}=0}^{a_s}\cdots\sum_{a_1=0}^{a_2}\binom{m+a_1}{a_1} \tag{5}$$

From (1) and (3′) we can obtain a further formula analogous to (4) with s and m interchanged.

In passing we remark that if $s=0$, and we require $n \geq 0$ in equation (1), the term in parentheses on the right side of (4) reduces to unity and we have the special relation

$$\sum_{a_M=0}^{m}\sum_{a_{M-1}=0}^{a_M}\cdots\sum_{a_1=0}^{a_2}1=\sum_{a_m=0}^{M}\sum_{a_{m-1}=0}^{a_m}\cdots\sum_{a_1=0}^{a_2}1, \tag{6}$$

a formula which can be interpreted through (1) to be equivalent to the well-known identity

$$\binom{m+M}{M} = \binom{M+m}{m}$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 216. Find the sum of $1(1!) + 2(2!) + 3(3!) + \ldots + n(n!)$

[*Submitted by M. S. Klamkin*]

Q 217. A father budgeted $6.00 to distribute equally among his children for spending money at the beach. When two young cousins joined the party and shared in the equal distribution, each child received 25 cents less than had been planned. How many children were in the party?

[*Submitted by C. W. Trigg*]

Q 218. Find an expression true for all n for the nth derivative of $\sin ax$.

[*Submitted by M. S. Klamkin*]

Q 219. Prove

$$\sum_{x=0}^{n} \binom{n}{x} (n-1)^{n-x} = n^n$$

[*Submitted by J. M. Howell*]

ANSWERS

A 219. For the solution consider the expansion of $[n-1+1]^n$.

A 218. $D^n \sin ax = a^n \sin(ax + \frac{n\pi}{2})$

A 217. Twenty-four quarters are involved. $24 = 2\times 12 = 3\times 8 = 4\times 6$. From these factorizations we pick two such that $q_1c_1 = q_2c_2$ and $q_1 + 1 = q_2$ while $c = c_2 + 2$. Thus there were 8 children in the party.

A 216. Since $n(n!) = (n+1-1)(n!) = (n+1)! - n!$ we have

$$\sum_{1}^{n} n(n!) = (n+1)! - 1$$

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 29. Factor 99,990,199. [*Submitted by Richard K. Guy*]

T 30. If 1/4 of 20 is 6, then what is 1/5 of 10? [*Submitted by C.W. Trigg*]

T 31. Show that $x^4\sqrt{3+x^2} - 4/3\,x^2(3+x^2)^{3/2} + 8/15\,(3+x^2)^{5/2}$ and $9\sqrt{3+x^2} - 2(3+x^2)^{3/2} + 1/5\,(3+x^2)^{5/2}$ are identically equal.

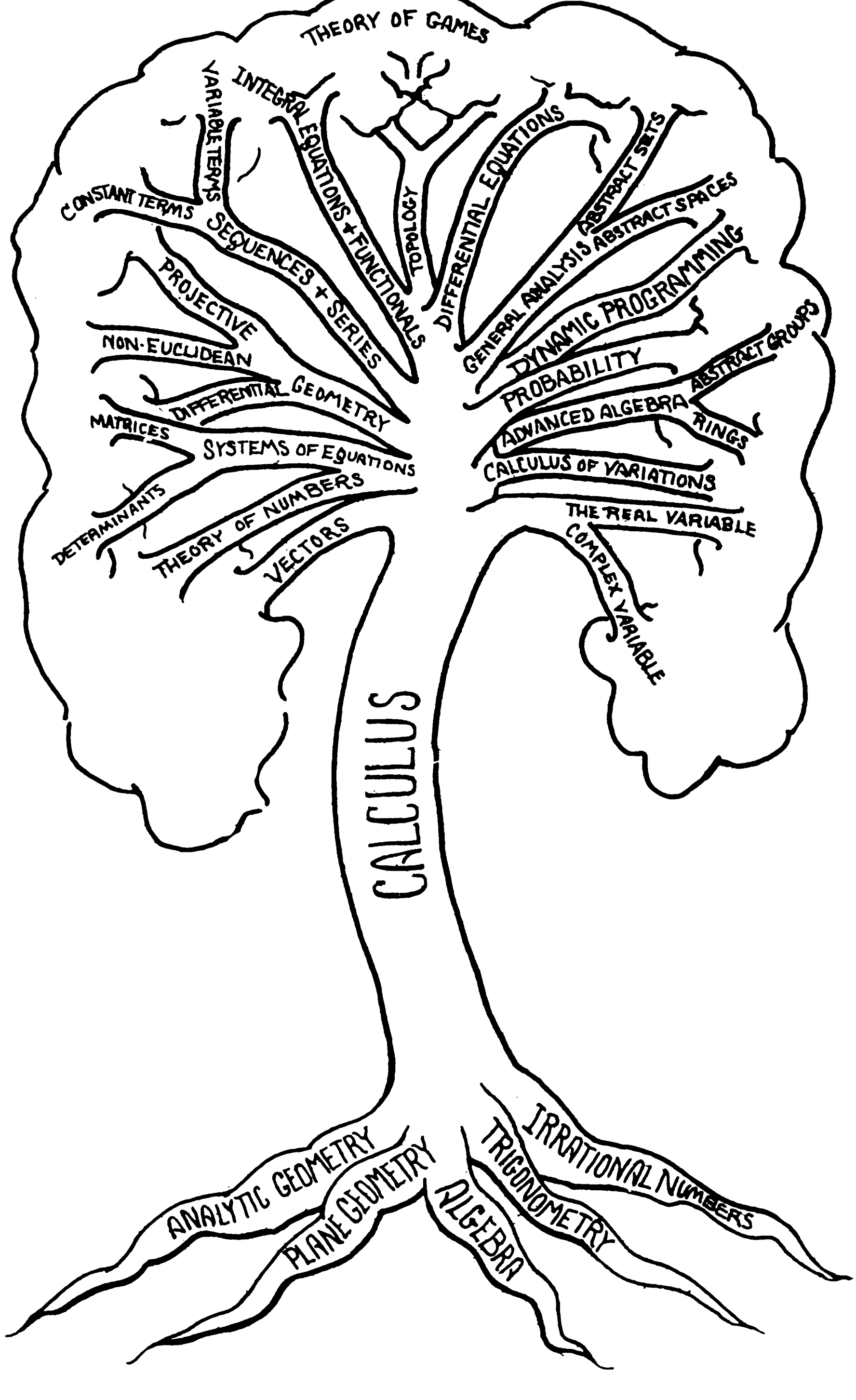
[*Submitted by Barney Bissinger's Calculus Class*]

SOLUTIONS

S 29. $99{,}990{,}199 = 10^8 - 9801 = (10{,}000)^2 - 99^2 = (9901)(10099)$

S 30. The computation is in the duodecimal scale so 1/5 of 10 is 2 2/5.

S 31. Each expression is an anti-derivative of $x^5\sqrt{3+x^2}$, one by parts and the other by substitution.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